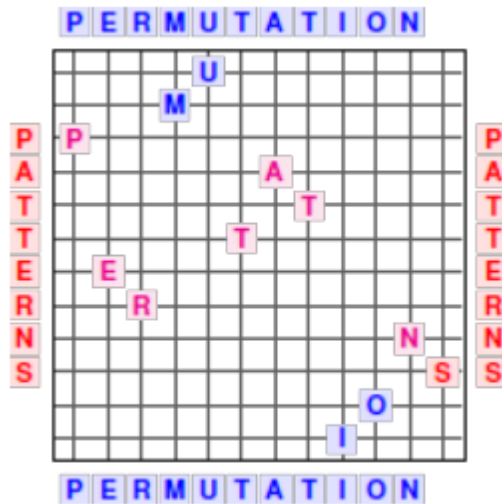




Permutation Patterns 2025

University of St Andrews

July 7–11 2025, St Andrews, Scotland



University of
St Andrews

WELCOME

Welcome to the University of St Andrews for Permutation Patterns 2025! We hope that you enjoy the conference, the town and Scotland.

We've included some suggestions of things to do and eat this week in the Local Information section of this book. Some of these you will be introduced to during the Wednesday walking tour of the town.

The main part of St Andrews consists of 3 streets (North, Market and South Street) and has interconnecting ones between them. St Andrews is mostly surrounded by the sea (the blue bit on the map) or golf courses (mostly the green bits on the map).

If you need anything during your stay, please do not hesitate to reach out to one of our local organisers (Christian Bean, Ruth Hoffmann, Joe Loughney, or Abi Ollson) who will be happy to help.

Whether you come from near or far, thank you for joining us in Scotland! We hope you will have a great week!

Ruth Hoffmann

(on behalf of the organising committees)

LOCAL INFO

Important Numbers

Ruth Hoffmann: +44 (0) 7857258771

McIntosh Hall Reception: +44 (0) 1334 467035

University Security and Response: +44 (0) 1334 468999

Accommodation/Conference and Events Team: +44 (0) 1334 463000

The hospital in St Andrews only has a minor injuries ward (open 8am to 5pm). The nearest Accident & Emergency unit is in Ninewells Hospital in Dundee.

You should call 999 if you have a medical emergency.

Breakfast/Lunches

If you are staying in McIntosh Hall, breakfast will be served 07:00 - 09:30 everyday in McIntosh Hall Dining Room.

Lunches (Mon - Fri) are provided by the University Conference Team and will be served in the open space outside the talk venue. On Friday they will provide packed lunches, which can be taken with you on your travels.

Conference Banquet

On Thursday evening the conference banquet will be held in Lower/Upper College Hall which is one of the buildings surrounding St Salvator's Quadrangle.

Local Restaurants

This is an extensive (but most likely not complete) list of places to eat, or drink in St Andrews.

We have separated the places into three (possibly subjective) cost categories (BF - Budget Friendly, MR - MidRange, TR - Top Range).

Restaurants

- Dune (TR, Seafood, has a Shellfish shack for a quick bite)
- Forgan's (MR, Scottish)
- Haar (TR, Seafood)

- Jahangir (BF, Indian)
- Little Italy (MR, Italian)
- Maisha (BF, Indian)
- Mitchell's (MR, Brunch/Sandwiches/Elevated Pub Grub)
- Mozza (BF, Scottish franchise, Italian/Pizza)
- Nando's (BF, UK franchise, Afro-Portuguese Chicken (but also some veggies))
- Pizza Express (BF, UK franchise, Italian/Pizza)
- Rogue (TR, Meat focussed)
- Tailend (BF, Fish & Chips/Seafood, there is a takeaway at the front, and a sit in restaurant at the back)
- Thai Popup (MR, Thai)
- The Adamson (TR, Brasserie)
- The Bothy (TR, Scottish)
- Seafood Ristorante (TR, Seafood)
- Steak Barn (MR/TR, Mostly Steak, some veggies)
- Zizzi's (BF, UK franchise, Italian)

Food that is more takeaway focussed but might still have some space to sit

- BlackHorn Burgers (MR, Burgers)
- Cormar's Fish&chips (BF, Fish&Chips)
- Dr Noodles (BF, Scottish franchise, Noodle Shop)
- Le Rendez-Vous (BF, European/Panini)
- Janetta's (MR, mostly ice cream, used to do other food)

Drinks (alcoholic and non-alcoholic)

Pubs which serve food (most do)

- Aikman's (BF)
- Brewdog (MR)
- Central Bar (BF)

- Greyfriar's Inn (MR)
- Hams Hame (MR)
- St Andrews Brewing Co. South Street (BF, local brewery)
- The Criterion (BF)
- The Rule (BF)
- The Saint Bar & Kitchen (MR)
- The Vic (MR)

Pubs which do not serve food

- Main Bar (Student Union)
- Molly Malones
- Whey Pat Tavern

Cafes (most will shut around 5pm)

- Black Sheep Coffee (UK franchise)
- Cafe in the Square
- Cafe Nero (UK franchise)
- Cheesy Toast Shack (recommend to pay the Seagull insurance)
- Costa (UK franchise)
- Cottage Kitchen
- Dook (they have a pop-up kitchen/van rotation which is usually open on week-ends)
- Fisher & Donaldson (famed for their fudge doughnut, Fife franchise)
- Greggs (UK franchise)
- Harbour Cafe
- Hatch
- Janettas (so good it gets named twice)
- North Point (where Will met Kate)
- Old Union Coffee Shop (University/Student Union run)

- Palompo's
- Pret a Manger (International franchise)
- Rector's Cafe (University/Student Union run)
- Starbucks (International franchise)
- Taste (St Andrews' (only?) speciality coffee shop)
- We are Zest Cafe

Parking

There is on-street parking all around St Andrews, but you have to pay for it. If you are staying in halls or a hotel check with the front desk if they can issue a permit for you.

Free (long-stay) parking is available in Petheram Bridge Carpark, which is located across the road from the Gateway building (where the conference is held).

Things to do

While in St Andrews there are a few things to see and do. Please note that a few of the local historical sites will be covered during Wednesday's walking tours and thus we will not go into details on them here.

(things that (most likely) will not be covered by the excursion on Wednesday)

- Botanical Garden's
- Bell Pettigrew Museum
- Eden Mill Distillery Tasting (I believe you can book a tasting in their St Andrews shop, or a dedicated pub/bar)
- East, West & Castle Sands (all beaches, West Sands was used for the filming of Chariots of Fire).
- Golf (too many to list)
- Himalaya Putting (<https://www.standrewsputtingclub.com/himalayas-putting-book-online/>)
- Kinburn Bowling Club <https://www.kinburnbowling.org.uk/> (they also do putting)
- The Pier

- St Andrews Aquarium
- St Andrews Museum

Things to do just a little bit out of St Andrews (within approx. 30mins walking distance)

- Craigtoun Country Park (has a little cafe as well)
- Fife Coastal Path <https://fifecoastandcountrysidetrust.co.uk/walks/fife-coastal-path/>
- Tentsmuir Forest *look at tide timetables and make sure to get back BEFORE the tide comes in*

Things nearby (recommend bus/public transport to get there)

- Eden Mill Distillery (only by booking! but they do offer a shuttle from/to St Andrews if a tasting is included)
- East Neuk (Crail for pottery; Anstruther for fish&chips; Pittenweem for fishing village vibes; the coastal path goes along there too)
- Kingsbarns Distillery
- Dundee (V&A Art Gallery/Museum; RRS Discovery; McManus Art Gallery&Museum; Breweries; Pubs; Food options)

MAPS

The University is very much integrated with the town, we thus have no well defined "campus".

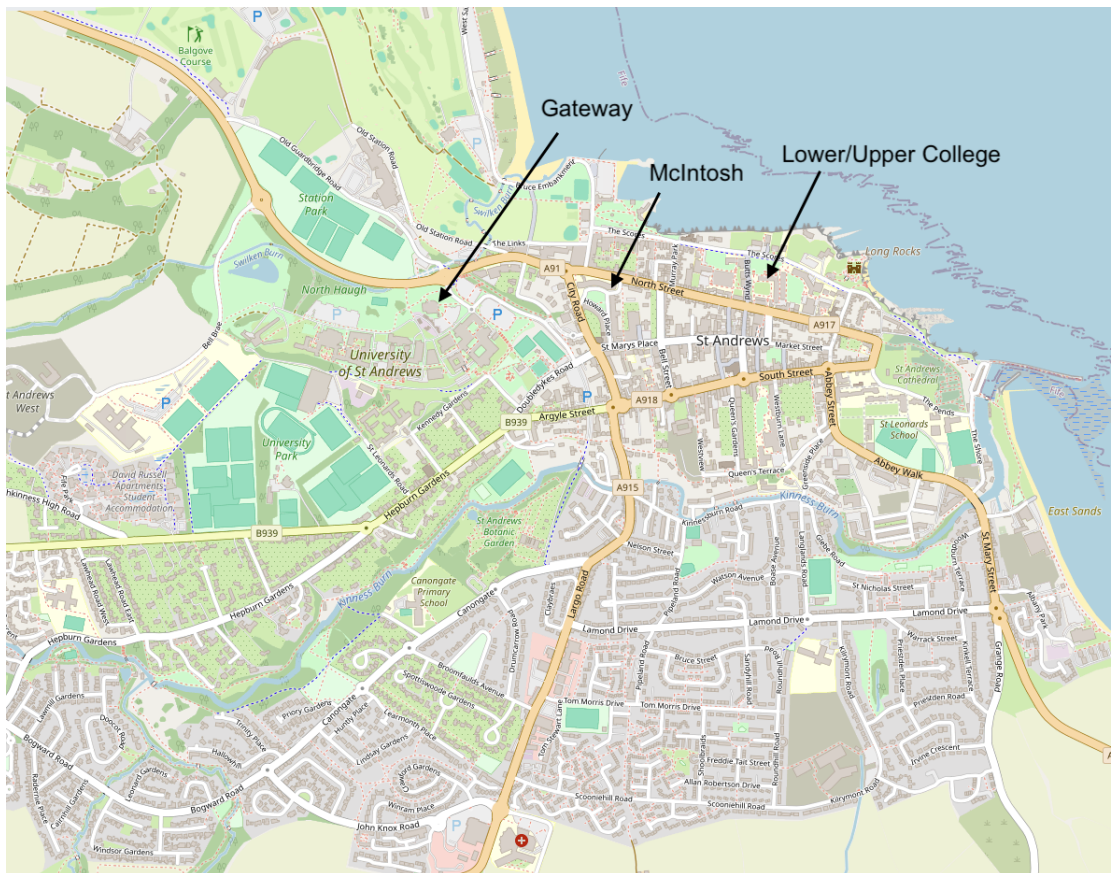
We have split the maps up into some smaller (more detailed) parts of St Andrews to help with the navigation. In general, the best marker to find your way around is to spot the sea and then make your way back away from it.

The University accommodation is in *McIntosh Hall*.

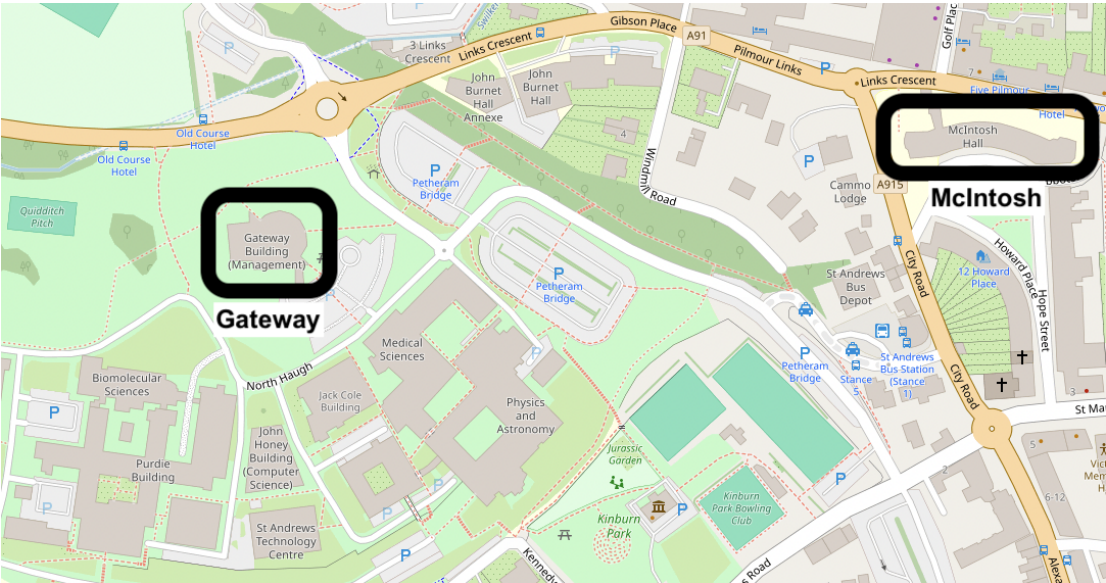
The conference and welcome reception are held in the *Gateway Building*.

The Banquet is in *Lower/Upper College*.

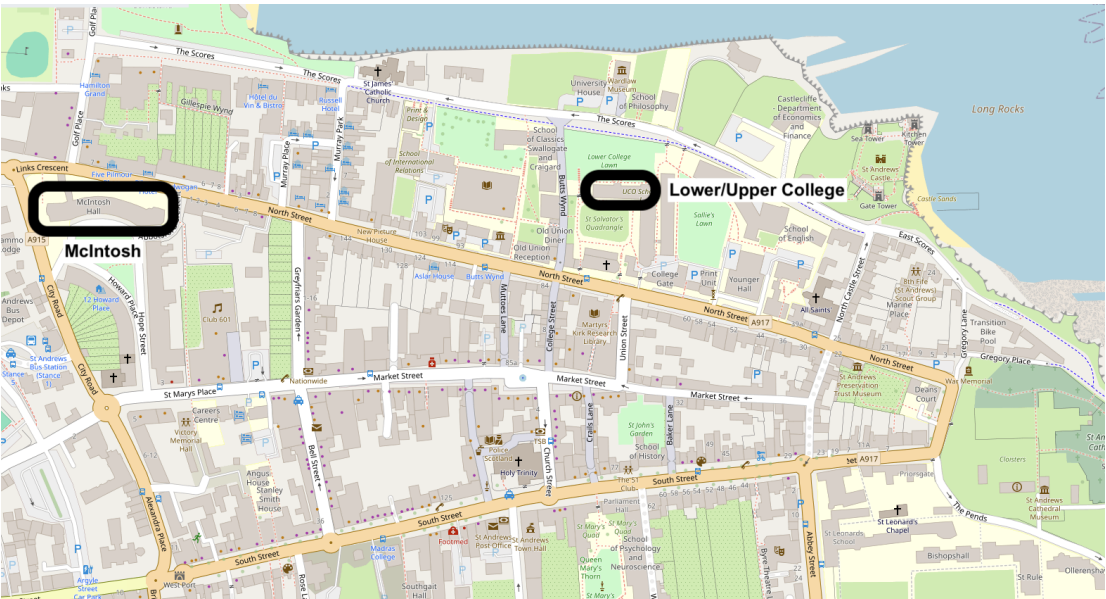
St Andrews



North Haugh & Gateway



The Town



CONFERENCE SOCIAL EVENTS

Monday, 7 July – Welcome Reception

A reception and poster session will be held in the foyer area of the Gateway building. Come and see the posters which will thereafter be kept there for the rest of the week. Nibbles and drinks will be provided and is included in the conference registration.

The reception will start shortly after the talks and last till 7pm.

Wednesday, 9 July – Walking Tours

The history of St Andrews is long and fascinating. The University and the town are strongly intertwined. So we have planned a walking tour of the town, which will cover the history of the town, the University and some golf. With main sights including the Cathedral, Castle, St Salvador's Chapel, and the Old Course. The walking tour will end at the University museum where we will get more insights into the University and its quirky traditions.

This is included in the cost of the registration for participants.

The walking tours will start at 2pm from the monument near the Golf Museum.

Thursday, 10 July – Conference Banquet

A conference banquet will be held in Lower/Upper College (on the ground floor) Hall. The banquet is included in the cost of the registration for participants.

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MONOTONE GRID CLASSES: LIMIT SHAPES AND ENUMERATION

David Bevan

University of Strathclyde

The (monotone) *grid class* $\text{Grid}(M)$ is a permutation class defined by a *gridding matrix* M whose entries are drawn from $\{\nearrow, \searrow, \square\}$. This matrix specifies the permitted shape for plots of permutations in the class. Each entry of M corresponds to a *cell* in an M -*gridding* of a permutation. If the entry is $\nearrow$, any points in the cell must form an increasing sequence; if the entry is $\searrow$, any points in the cell must form a decreasing sequence; if the entry is $\square$, then the cell must be empty. With a minor abuse of terminology, we often refer to the matrix entries themselves as cells, calling $\square$ entries *blank* cells, and $\nearrow$ and $\searrow$ entries *non-blank* cells. A permutation can have more than one M -gridding. See Figure 1 for an example.

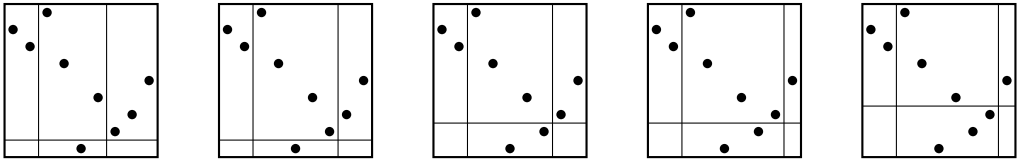


Figure 1: The five $\begin{smallmatrix} \searrow & \searrow \\ \searrow & \searrow \end{smallmatrix}$ -griddings of 879614235

In this talk, we will consider the following questions:

- What does a typical large permutation in a given monotone grid class look like? That is, what is the *limit shape* of the class?
- What is known (and what is not known) about the enumeration (exact and asymptotic) of monotone grid classes?

The presentation will include several open questions, of varying perceived difficulty.

THE DISTRIBUTION OF POINTS BETWEEN CELLS

The *gridded class*, consisting of all M -gridded permutations, is denoted by $\text{Grid}^\#(M)$. If $\sigma^\# \in \text{Grid}^\#(M)$ then $\sigma^\#_{(i,j)}$ denotes the number of points of $\sigma^\#$ in cell (i,j) .

A grid class $\text{Grid}(M)$ is *connected* if it has a connected *cell graph*, which is the graph whose vertices are the non-blank cells of M , and in which two vertices are adjacent if they share a row or column and every cell between them is blank. For example, the (connected) cell graph of $\begin{smallmatrix} \searrow & \searrow & \searrow \\ \searrow & \searrow & \searrow \end{smallmatrix}$ is

Key to our results is determining the proportion of points that occur in each cell in a typical large M -gridded permutation. If $\text{Grid}(M)$ is connected, then there is a unique, explicitly computable, real M -*distribution* matrix $\Gamma_M = (\gamma_{i,j})$ such that for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}[\max_{i,j} |\sigma^\#_{(i,j)}/n - \gamma_{i,j}| \leq \varepsilon] = 1,$$

where, for each n , the gridded permutation $\sigma^\#$ is drawn uniformly from $\text{Grid}_n^\#(M)$.

For example, if $M_T = \begin{bmatrix} \text{X} & \text{X} & \text{X} \\ \text{X} & \text{X} & \text{X} \\ \text{X} & \text{X} & \text{X} \end{bmatrix}$, then $\Gamma_{M_T} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 \end{pmatrix}$.

LIMIT SHAPES

Given a permutation class, it may be that almost all large permutations in the class have the “same shape”. To formalise this idea, we make use of certain probability measures which act as analytic limits of sequences of permutations. A *permuton* is a probability measure μ on the unit square $[0,1]^2$ with uniform marginals. That is, $\mu([a,b] \times [0,1]) = \mu([0,1] \times [a,b]) = b - a$, for every $0 \leq a \leq b \leq 1$. There is a natural topology on the space of permutons obtained by restricting the weak topology on probability measures.

A permuton μ_σ can be associated with an n -permutation σ by taking its plot, scaling it into the unit square, and replacing each point with a square of area $1/n^2$ and density n , as illustrated in Figure 2.



Figure 2: A plot of the permutation $\pi = 314592687$, and a picture of the permuton μ_π

Given a permutation class $\mathcal{C}$, let σ_n be an n -permutation drawn uniformly at random from $\mathcal{C}_n$. If the sequence of random permutons $(\mu_{\sigma_n})_{n \geq 1}$ converges in distribution for the weak topology to some (possibly random) permuton μ , then we say that μ is the *limit shape* of $\mathcal{C}$.

Every connected grid class has a *deterministic* limit shape whose support consists of oblique line segments. For example, the limit shape for $\text{Grid}(M_T)$ is shown in Figure 3, the top row and central column both having width $\frac{2}{3}$.

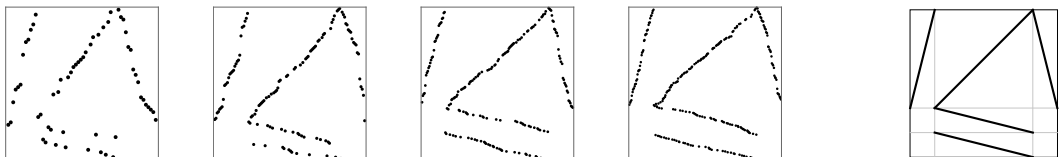


Figure 3: Plots of permutations of length 60, 120, 180 and 240 in $\text{Grid}(M_T)$, and the limit shape of the class

ENUMERATION

Any monotone grid class whose cell graph is *acyclic* has a rational generating function. However, this result is nonconstructive, and no effective procedure is known to establish the generating function of an arbitrary acyclic class. On the other hand, acyclic *gridded* classes also have rational generating functions, which are straightforward to compute. This provides us with a strategy for determining the *asymptotic*

enumeration of a connected acyclic grid class $\text{Grid}(M)$:

1. Find Γ_M , as described above.
2. For each $\ell \geq 1$, determine how a typical large M -gridded permutation $\sigma^\#$ must be structured so that its underlying permutation σ has ℓ distinct M -griddings. (Rare atypical structures can be ignored.) This involves analysing how the row and column dividers may move, as in Figure 1.
3. Let $\sigma_n^\#$ be a gridded permutation drawn uniformly at random from $\text{Grid}_n^\#(M)$, and let σ_n be its underlying permutation. By combining steps 1 and 2, calculate, for each $\ell \geq 1$, the asymptotic probability

$$P_\ell := \lim_{n \rightarrow \infty} \mathbb{P}[\sigma_n \text{ has exactly } \ell \text{ distinct } M\text{-griddings}].$$

4. From its generating function, determine the asymptotic enumeration of the corresponding gridded class: $|\text{Grid}_n^\#(M)| \sim \theta^\# g^n$, where g is the exponential growth rate of the class, and $\theta^\#$ is a constant.
5. Let $\kappa_M := \sum_{\ell \geq 1} P_\ell / \ell = \lim_{n \rightarrow \infty} \frac{|\text{Grid}_n(M)|}{|\text{Grid}_n^\#(M)|}$. Then $|\text{Grid}_n(M)| \sim \kappa_M \theta^\# g^n$.

For example, we have the following for $\text{Grid}(M_\top)$:

1. $\Gamma_{M_\top}$ is given at the top of the previous page.
2. Depending on the position of points relative to the row and column dividers, a typical large $M_\top$ -gridded permutation $\sigma^\#$ may be such that its underlying permutation σ has either 1, 2 or 4 distinct griddings.
3. The asymptotic probabilities are $P_1 = \frac{1}{9}$, $P_2 = \frac{4}{9}$ and $P_4 = \frac{4}{9}$.
4. The gridded class has generating function

$$F_{M_\top}^\#(z) := \sum_{n \geq 0} |\text{Grid}_n^\#(M_\top)| z^n = \frac{1}{1 - 5z + 4z^2}.$$

Thus, $|\text{Grid}_n^\#(M_\top)| \sim \frac{4}{3} \times 4^n$.

5. The correction factor $\kappa_{M_\top} = \frac{4}{9}$. Therefore, $|\text{Grid}_n(M_\top)| \sim \frac{16}{27} \times 4^n$.

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This talk is based on joint work with Slim Kammoun and Einar Steingrímsson

Definition 1. Fix $k, j \in \mathbb{N}$, $\pi \in S_k$, and $m \in \{1, \dots, k\}$. We say that a permutation σ is m -packing j occurrences of the consecutive permutation pattern π if σ contains occurrences of π starting in positions $\{1, 1 + k - m, 1 + 2(k - m), \dots, 1 + (j - 1)(k - m)\}$.

In other words, given a fixed consecutive pattern $\pi \in S_k$, we consider permutations $\sigma \in S_n$ for $n \geq k$ (in fact, $n = k + (j - 1)(k - m)$) in which π is guaranteed to occur at regular intervals. (Note that Definition 1 does not preclude additional occurrences, outside of the regularly-occurring ones.) For example, the permutation 14263758 is 2-packing 3 occurrences of 1324: namely, 1426, 2637 and 3758.

At first sight, the definition may seem familiar. Without explicitly keeping track of the positions of the occurrences, and instead considering permutations $\sigma \in S_n$ such that every position $i \in \{1, \dots, n\}$ is included in some occurrence of π , one can interpret σ as a ‘cluster’, in the sense of the cluster method of Goulden and Jackson. (See e.g. [1, 12] for some enumerative consequences of this point of view.)

Moving beyond clusters and tracking the *positions* of occurrences of a consecutive pattern is also natural: it generalizes the notion of the descent set of a permutation. Special cases of the construction of Definition 1 have appeared in the combinatorics literature [7, 8, 9, 10], but, to the best of our knowledge, these objects have never been considered in the generality presented here. Furthermore, drawing a permutation σ at random, the starting positions occurrences of a given pattern π gives rise to a *point process* (a natural generalization of the descent-tracking point process in [2]). When the permutation is drawn uniformly at random, describing the point process, in terms of its *correlation functions*, reduces to counting permutations σ containing occurrences of π in prescribed positions. (And when the permutation is not uniform, but is drawn according to some conjugacy-invariant law, we show that the conclusion is very similar – we shall touch upon this briefly in our lecture as well.)

More to the point, in addition to distinguishing the positions of the occurrences of π , we also require these to occur at regular intervals, with prescribed ‘overlaps’ m . This regularity is key. In particular, the inequalities satisfied by the elements of any such permutation σ give rise to posets (not to be confused with the posets in [11]) that themselves exhibit regularity. For a given pattern π , the number of permutations m -packing j occurrences of π equals the number of linear extensions of a poset defined in terms of π , m and j . An element in such a poset can cover (and be covered by) at most two elements, and there are further limitations on their structure.

Taken together, these definitions lead us to the following two classes of results, which we will discuss in this talk:

- The posets arising from packings of consecutive permutation patterns give rise to interesting classification problems. Furthermore, the poset point of view,

combined with a probabilistic perspective, opens up a broader class of enumerative techniques and yields **new enumerative results** for permutations packing consecutive patterns. These take the form of explicit integer sequences, generating functions, integrals, or matrix products, and extend well beyond the initial exploration in [3].

- The regularity of the construction, for reasons that will be discussed in the lecture, suggests a link to moment sequences. Acting on an intuition originally presented in [3], we introduce **new general families of combinatorial moment sequences** associated with consecutive permutation patterns – and conjecture many more. In particular, Definition 1 provides a setting in which analogues of the recent ‘positivity’ questions associated with permutation patterns (see e.g. [4, 5, 6]) can be systematically studied and answered for *consecutive* permutation patterns.

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AVOIDING A SINGLE PATTERN IN THE BRUHAT ORDER

Ben Adenbaum

Florida Gulf Coast University

The Bruhat order on $\mathfrak{S}_n$ is a classical partial order that naturally arises in the study of $\mathfrak{S}_n$ as a Coxeter group. In particular pattern avoidance naturally arises in classifying elements whose principal order ideals are isomorphic to subsets under inclusion as was shown by Tenner in [5]. Alternatively it was shown by Barcucci, Bernini, Ferrari, and Poneti in [1] that when considering $Av_n(312)$ as a subposet of the Bruhat order the resulting partial order is the distributive lattice corresponding to Dyck Paths partially ordered by inclusion.

We attempt to study the subposets of the Bruhat order corresponding to $Av_n(\sigma)$ for general $\sigma \in \mathfrak{S}_k$. In particular we focus on the case where $\sigma = 2143$. Our main results concerning these posets can be summarized as follows.

Proposition 1. *If $\sigma \notin \{1, 12, 21, 132, 213, 231, 312\}$ then $Av_n(\sigma)$ is not a lattice.*

Theorem 2. *There is an order preserving map $\Phi : Av_n(2143) \rightarrow Av_n(132)$ that preserves the number of inversions satisfying $\Phi^2 = \text{id}$.*

A generalization of permutations are *alternating sign matrices*.

Definition 3 (Alternating Sign Matrices). An *alternating sign matrix* is a square matrix where each entry is either 0, 1, or -1 , every row and column sum is 1, and the non-zero entries in each row alternate in sign between 1 and -1 . Denote the set of all $n \times n$ alternating sign matrices by ASM_n .

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Figure 4: The alternating sign matrices of ASM_3

One way of realizing this partial order is via the *corner sum matrix* of $\pi \in \mathfrak{S}_n$. The alternating sign matrices arise naturally in the study of the Bruhat order. They also can be partially ordered via the corner sum matrix order and this order is the lattice completion of the Bruhat order.

Definition 4. Let $A \in ASM_n$. The *corner sum matrix* of σ , denoted by $csm(\sigma)$, is the $n \times n$ matrix where $csm(i, j) = \sum_{l \leq i, k \leq j} a_{l, k}$. The partial order on alternating sign matrices is the partial order defined by $A \geq B$ if for all $1 \leq i, j \leq n$, $csm(A)_{i, j} \leq csm(B)_{i, j}$.

In particular there is a notion of what it means for an alternating sign matrix to contain a pattern.

Definition 5 ([4]). An alternating sign matrix A contains an occurrence of a permutation π if there is a submatrix D where $D_{ij} = 1$ if $\pi(i) = j$. Denote by $ASMAv_n(\pi)$ the set of $n \times n$ alternating sign matrices which do not contain an occurrence of π .

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Figure 5: The corner sum matrices of ASM_3

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & \color{red}{1} & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & \color{red}{1} & 0 & 0 \\ 0 & 0 & \color{red}{1} & 0 & 0 & 0 \end{bmatrix}$$

Figure 6: A 6×6 alternating sign matrix with an occurrence of 132 highlighted.

In recent years there has been a surge of activity concerning alternating sign matrices avoiding a permutation pattern [2, 3], which was first studied in [4]. However understanding the partial order structure of alternating sign matrices avoiding a pattern does not seem to be studied. We show that the restriction of the Bruhat order to $Av_n(312)$ of [1] can be generalized to the setting of alternating sign matrices avoiding a single nonmonotonic pattern of length 3 as follows.

Theorem 6. *The corner sum matrix partial order on alternating sign matrices restricted to $AvASM_n(132)$ is isomorphic to the partial order on Schröder paths ordered by reverse containment.*

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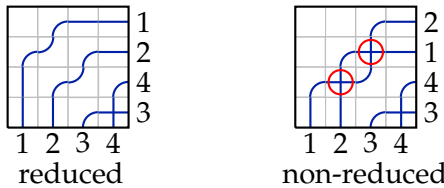
This talk is based on joint work with Adam Gregory

We give a pattern-avoidance characterization of a certain set of permutations arising from Schubert calculus, solving a problem of Weigandt [9]. We begin by introducing the definitions needed to state our result. Afterwards, we provide context and discuss our motivation for finding such a characterization.

A *bumpless pipe dream* (BPD) of size n is a tiling of an $n \times n$ grid with tiles of the form



such that n pipes enter from the bottom and exit to the right [3]. Labeling the pipes in increasing order from left-to-right as they enter, one obtains an *associated permutation* by reading the pipe labels from top-to-bottom as they exit, treating multiple crossings between a pair of pipes as turns if such an occurrence exists. A BPD is *reduced* if no two pipes cross more than once and *non-reduced* otherwise.

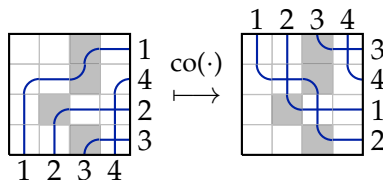


Write $\text{BPD}(w)$ for the bumpless pipe dreams with associated permutation w . We refer to the  and  tiles as *elbows*. Every BPD is completely determined by the location of its elbows [8]. Note that BPDs are in bijection with alternating sign matrices (ASM) by replacing each  with a 1, each  with a -1 , and all other tiles with a 0 [8]. In recent work, Weigandt defined a co-BPD object corresponding to each BPD.

Definition 1 ([9]). For a given bumpless pipe dream B , its corresponding *co-bumpless pipe dream* $\text{co}(B)$ is defined by exchanging tiles as follows:

$$\begin{array}{c} \text{bottom-left elbow} \longleftrightarrow \text{top-left elbow}, \\ \text{top-left elbow} \longleftrightarrow \text{bottom-left elbow}, \\ \text{vertical line} \longleftrightarrow \text{square}, \\ \text{horizontal line} \longleftrightarrow \text{cross}. \end{array}$$

In other words, the locations of the elbows are the same, but now pipes enter from the top instead of the bottom while still exiting to the right. Labeling pipes in increasing order from left-to-right as they enter, $\text{co}(B)$ *traces out* a permutation by reading the pipe labels from top-to-bottom as they exit, treating any multiple crossings as turns. The permutation associated to B may be different from the one $\text{co}(B)$ traces out.



A co-BPD is *reduced* if no two pipes cross more than once and *non-reduced* otherwise. Notice that $\text{co}(\cdot)$ does not necessarily preserve reducedness as seen in the example above. We refer to the set $\{\text{co}(B) : B \in \text{BPD}(w)\}$ as the co-BPDs for w .

We now state our main result.

Theorem 2. *All co-BPDs for a permutation w are reduced if and only if w avoids the seven patterns 1423, 12543, 13254, 25143, 215643, 216543, and 241653.*

Motivation

We now provide context to motivate our result.

Schubert polynomials $\mathfrak{S}_w$ are a celebrated family of functions indexed by permutations $w \in S_n$. They generalize Schur functions and represent cohomology classes of Schubert varieties in the complete flag variety [4]. Grothendieck polynomials $\mathfrak{G}_w$ are K -theoretic generalizations of Schubert polynomials [5].

There are several known combinatorial formulas for $\mathfrak{S}_w$. One of the earliest is the *pipe dream* (PD) model of Billey–Bergeron [1] that extends to $\mathfrak{S}_w$ by Fomin–Kirillov [2]. A more recent one is the *bumpless pipe dream* (BPD) model of Lam–Lee–Shimozono [3] that was also extended to $\mathfrak{G}_w$ by Weigandt [8] using the connection to ASMs.

For a permutation w , let $\text{bpd}(w) \subseteq \text{BPD}(w)$ be the subset of bumpless pipe dreams for w that are reduced. For $B \in \text{BPD}(w)$, define $\text{blank}(B)$ to be the set of locations of each $\square$ tile and $\text{jay}(B)$ to be the set of locations of each $\blacksquare$ tile. We can then define $\mathfrak{S}_w$ and $\mathfrak{G}_w$ as sums over bumpless pipe dreams given the following monomial weighting:

$$\mathfrak{S}_w = \sum_{B \in \text{bpd}(w)} \prod_{(i,j) \in \text{blank}(B)} x_i ; \quad ([3])$$

$$\mathfrak{G}_w = \sum_{B \in \text{BPD}(w)} (-1)^{\ell(w)} \left(\prod_{(i,j) \in \text{blank}(B)} -x_i \right) \left(\prod_{(i,j) \in \text{jay}(B)} 1 - x_i \right). \quad ([8])$$

Both families of polynomials are bases for the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$. Their change of bases formulas have been studied combinatorially by both Lenart [7] and Lascoux [6] using the pipe dream model. Weigandt introduced these co-BPD objects to provide analogous formulas using bumpless pipe dreams.

Theorem 3 ([9]). *Let $a_{w,v} = \#\{B \in \text{BPD}(w) : \text{co}(B) \text{ reduced \& traces out } v\}$. Then*

$$\mathfrak{G}_w = \sum_v (-1)^{\ell(v) - \ell(w)} a_{w,v} \cdot \mathfrak{S}_v .$$

Theorem 4 ([9]). *Let $b_{w,v} = \#\{B \in \text{bpd}(w) : \text{co}(B) \text{ traces out } v\}$. Then*

$$\mathfrak{S}_w = \sum_v b_{w,v} \cdot \mathfrak{G}_v .$$

Theorem 3 naturally leads to asking which permutations have $\sum_v a_{w,v} = |\text{BPD}(w)|$, which is answered by our main result Theorem 2. It is interesting to note that the solution in the BPD setting is more complicated than the corresponding one in the pipe dream setting, where it is simply 132-avoiding (i.e., dominant) permutations.

Future Directions

For a given w , little is known about the possible v that can contribute to $a_{w,v}$. However, as a corollary of our main result we can at least say the following.

Corollary 5. *If B is a non-reduced BPD, then $\text{co}(B)$ traces a permutation containing the reverse of at least one of the patterns from Theorem 2.*

On the other hand, we wonder what one can say about the set of permutations *containing* one of the patterns 1423, 12543, 13254, 25143, 215643, 216543, or 241653 from Theorem 2. Notably, is there an alternate description of this set of permutations using other notions of pattern containment?

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$\top$ -AVOIDING RECTANGULATIONS AND $I(010, 101, 120, 201) \cong I(011, 201)$

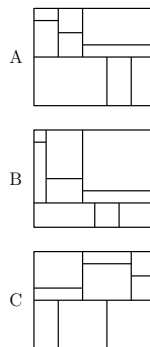
Andrei Asinowski¹

University of Klagenfurt

This talk is based on joint work with Michaela Polley² (Dartmouth College).

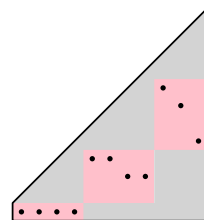
Abstract. We show that $\top$ -avoiding rectangulations are in bijection with several classes of inversion sequences, among them $I(010, 101, 120, 201)$ and $I(011, 201)$. This provides a proof for the conjecture [7] that $I(011, 201)$ is enumerated by A279555, and finalizes the classification of classes of inversion sequences, determined by four patterns of length 3, which are enumerated by this sequence [3].

Rectangulations. A *rectangulation* of size n is a partition of a rectangle R into n rectangles. Rectangulations form a combinatorial class under either of two kinds of equivalence: *weak equivalence* which preserves segment-to-rectangle adjacencies, and *strong equivalence* which (also) preserves rectangle-to-rectangle adjacencies. In the image, all three rectangulations are weakly equivalent, but only A and B are strongly equivalent. Equivalence classes are, accordingly, referred to as *weak rectangulations* and *strong rectangulations*. See [1] for more details, summary of basic results, and recent developments.



Patterns in Rectangulations implicitly appeared in several early contributions, but systematic investigation only started recently; see for example [5]. In [2], we presented enumerative results for rectangulations that avoid the pattern $\top$ and its rotations, in all possible combinations. In the present contribution, we further investigate $\top$ -avoiding strong rectangulations — which, surprisingly, leads to a solution of a conjecture concerning enumeration of avoidance classes of inversion sequences.

Inversion sequences enumerated by OEIS A279555. The sequence A279555 was contributed by Martinez and Savage as the enumerating sequence of two classes of inversion sequences, $I(010, 110, 120, 210)$ and $I(010, 100, 120, 210)$ [4]. We also consider the avoidance class $I(010, 101, 120, 201)$. It is very easy to show that it is equinumerous with both former classes, but it has a simpler structure as shown in the image: all the points of the plot of such inversion sequences are contained in an ascending sequence of rectangular areas (in pink), the points within every rectangular area are weakly descending, and one of them lies in the top-left corner of the area.



In [7, Conjecture 8.3], Yan and Lin conjectured that $I(011, 201)$ and $(011, 210)$ are also enumerated by A279555. Pantone [6, Section 6.5] provided strong support for this

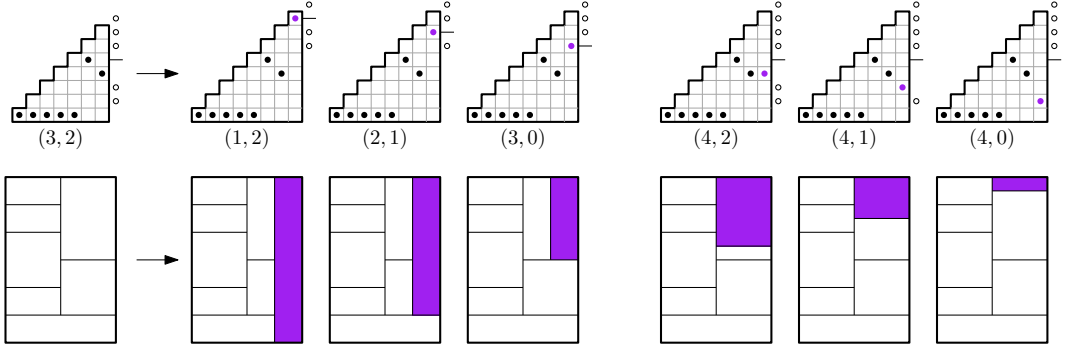
¹Research of Andrei Asinowski was funded by the Austrian Science Fund (FWF) [10.55776/P32731].

²Research of Michaela Polley was funded by a Fulbright-Austrian Marshall Plan Foundation Award For Research In Science And Technology.

conjecture by computing 500 terms of $I(011,201)$ and $I(010,100,120,210)$. Finally, in [3, Table 1, Case 166], Callan and Mansour gave nine avoidance classes of inversion sequences determined by four patterns of length 3: four of them were confirmed to be enumerated by [A279555](#) at the time of writing, while for five others (two of them were in fact alternative descriptions of $I(011,201)$ and $(011,210)$) it was still conjectured. In this contribution, we provide proofs for these conjectures.

Bijections between $\top$ -avoiding rectangulations and two classes of inversion sequences: $I(010,101,120,201)$ and $I(011,201)$. In our proofs, we use generating trees for $I(010,100,120,210)$ and for $I(011,201)$, given by Pantone [6, Section 6.5]. We show that both these trees generate the $\top$ -avoiding rectangulations — hence, these classes of inversion sequences are equinumerous. In accordance with the remark above, for the former class we use the equivalent class $I(010,101,120,201)$.

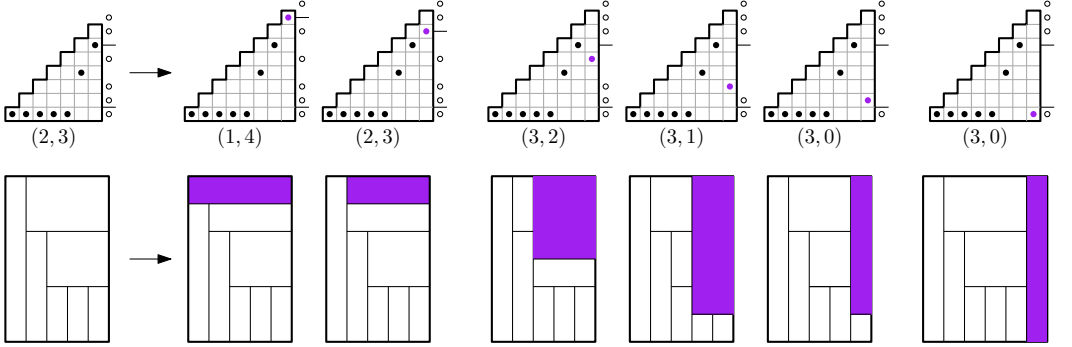
The generating tree that applies to $I(010,101,120,201)$ and to $\top$ -avoiding rectangulations, has root $(1,0)$ and succession rules $(k, \ell) \rightarrow (1, k-1), (2, k-2), \dots, (k, 0); (k+1, 0), (k+1, 1), \dots, (k+1, \ell)$. For $e \in I(010,101,120,201)$, k is the *bounce* — the difference between the size of e and its maximum value M ; and ℓ is the number of values smaller than M that can be inserted at the end of e to produce a new sequence in the class. For a $\top$ -avoiding rectangulation, k is the number of rectangles that touch the right side of R ; and ℓ is the number of horizontal segments whose right endpoint touch the top-right rectangle. Inserting a new point at the end of e corresponds to inserting a new top-right rectangle into the corresponding rectangulation:



For the second construction, we use $\perp$ -avoiding (instead of $\top$ -avoiding) rectangulations. The generating tree that applies to $I(011,201)$ and to $\top$ -avoiding rectangulations, has root $(1,0)$ and succession rules $(k, \ell) \rightarrow (1, k+\ell-1), (2, k+\ell-2), \dots, (k, \ell); (k+1, 0), (k+1, 1), \dots, (k+1, \ell-1); (k+1, 0)$. For $e \in I(011,201)$, k and ℓ have the same meaning as above with an additional condition that ℓ refers to *non-zero* admissible values. For a $\perp$ -avoiding rectangulation, k is the number of rectangles that touch the top side of R ; and ℓ is the number of $\top$ joints whose horizontal segment touches the right side of R . As in the first construction, inserting a new point at the end of e corresponds to inserting a new top-right rectangle into the corresponding rectangulation (see the image on the next page).

The constructions sketched above yield bijections between both classes of inversion sequences and $\top$ -avoiding rectangulations — hence, $I(010,101,120,201)$ and $I(011,201)$

are in bijection. Moreover, the numbers of rectangles that touch top, bottom, right, and left sides of R correspond to certain statistics in inversion sequences. Therefore, we obtain not only a bijection between two classes of inversion sequences, but also the following correspondence between quadruples of statistics:



Theorem. For every $n \geq 1$, we have $|I_n(010, 101, 120, 201)| = |I_n(011, 201)|$. Moreover, the quadruple of statistics (a, b, c, d) for $I_n(010, 101, 120, 201)$, where a is the number of elements that satisfy $e_j = 0$, b is the number of left-to-right-maxima, c is the bounce, and d is the number of elements that satisfy $e_j = j - 1$, matches the quadruple of statistics (x, y, z, t) for $I_n(011, 201)$, where x is the number elements that satisfy $e_j = j - 1$, y is the number of elements that satisfy $e_j = 0$, z is the number of right-left-minima, and t is the bounce.

We also construct direct bijections which align with the generating trees given above.

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Introduction

The notion of an interval poset for a permutation, initially introduced by Tenner in [5], effectively represents all intervals within a permutation and their inclusion relationships comprehensively.

Definition 1. The *interval poset* of a permutation $\pi \in \mathcal{S}_n$ is the poset $P(\pi)$ whose elements are the non-empty intervals of π ; the order is defined by set inclusion and the minimal elements are the intervals of size 1.

Bouvel, Cioni and Izart [3], provided a formula for the number of interval posets with n minimal elements, sequence A348479 in [13]. Tenner enumerated binary interval posets. These are exactly the interval posets of the separable permutations. These posets are counted by the small Schröder numbers. She also enumerated binary tree interval posets by $2C_{n-1}$ where C_n is the n -th Catalan number. Then, Bouvel, Cioni and Izart in [3] enumerated the tree interval posets (sequence A054515 in [13]). Interestingly enough, this sequence also enumerates the number of ways to place non-crossing diagonals in a convex $(n+1)$ -gon such that no quadrilaterals are created.

In this study (see the full paper in [2]), we propose a different view on interval posets by introducing a simple bijection between the set of tree interval posets and the set of dissections of convex $(n+1)$ -gons, satisfying the conditions listed above. We also use this bijection to enumerate interval posets of various subsets of permutations by using a wider set of polygons.

Interval posets and convex polygons

We identify a convex polygon with its set of vertices and denote a diagonal or an outer edge of the polygon from vertex i to vertex j by (i, j) .

Let P be a convex n - polygon. A *dissection* of P is a decomposition of it into finitely many sub-polygons.

Definition 2. A dissection of an n - gon will be called *diagonally framed* if for each two crossing diagonals, all their vertices are connected to each other. Explicitly, if (a, c) and (b, d) are two crossing diagonals, then the diagonals or outer edges $(a, b), (b, c), (c, d), (d, a)$ must also exist (see Fig. 7 for an example).

We define a bijection between the set of interval posets with n minimal elements and the set of diagonally framed dissections of convex $(n+1)$ - gon without quadrilaterals as follows:

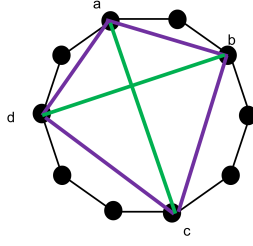


Figure 7: A 10 - gon with crossing diagonals (a, c) and (b, d) and their "frame".

Let P be the interval poset of some $\pi \in \mathcal{S}_n$. We set $\Phi(P)$ to be the convex $(n + 1)$ -gon whose set of diagonals is

$$\{(a, b + 1) \mid [a, b] \text{ is a non-minimal element of } P\},$$

i.e. to each interval of the form $[a, b]$ corresponds a diagonal $(a, b + 1)$ in $\Phi(P)$; note that singleton intervals correspond to outer edges in the polygon (see Figure 8 for an example).

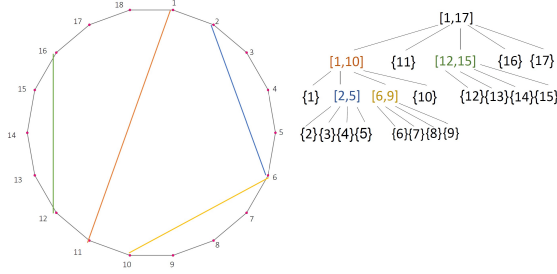


Figure 8: Right: the interval poset P . Left: the polygon $\Phi(P)$

Using the above bijection we proved the following results.

Theorem 3. *The number of interval posets with n minimal elements is equal to the number of diagonally framed dissections of the convex $(n + 1)$ -gon such that no quadrilaterals are present*

Theorem 4. *The number of tree interval posets with n minimal elements is equal to the number of non-crossing dissections of the convex $(n + 1)$ -gon such that no quadrilaterals are present.*

In [1], the current authors introduced the notion of block-wise simple permutations. We cite here the definition:

Definition 5. A permutation $\pi \in \mathcal{S}_n$ is called *block-wise simple* if it has no interval of the form $p_1 \oplus p_2$ or $p_1 \ominus p_2$, where $\oplus$ and $\ominus$ stand for direct and skew sums of permutations respectively.

Theorem 6. *The number of interval posets that represent a block-wise simple permutation of order n is equal to the number of ways to place non-crossing diagonals in a convex $(n + 1)$ -gon such that no triangles or quadrilaterals are present*

The number of separable permutations of order n is known to be the (big) Schröder number (sequence A006318 in the OEIS [13]) . In [5], Theorem 6.3, the author proved that each interval poset of a separable permutation corresponds to exactly two separable permutations, from which she concluded that the number of such interval posets is half this number. This last number is known as counting also the dissections of the $n + 1$ - gon such that crossing diagonals are prohibited. (see [6]). Here we present a bijection that proves this connection in a combinatorial way.

Theorem 7. *There is a bijection between the interval posets corresponding to separable permutations of order n and the set of dissections of the $n + 1$ - gon such that crossing diagonals are prohibited.*

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SOME INTEGER VALUES IN THE SPECTRA OF THE CAYLEY GRAPHS OF COLORED PERMUTATIONS GENERATED BY PREFIX REVERSALS

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This talk is based on joint work with Charles Buehrle

If G is a graph with adjacency matrix $A(G)$, then we refer to the set of eigenvalues of $A(G)$ along with their respective multiplicities as the *spectrum* of G . Cayley graphs are fundamental structures in algebra, and understanding their spectral properties sheds light on their structure. Some of these graphs have extremely nice spectra. For example, consider the so-called *Transposition graphs*, where two permutations π_1 and π_2 are connected by an edge if and only if there is a transposition t such that $t\pi_1 = \pi_2$. The transposition graph is *integral*, meaning that all of its eigenvalues are integers.

Another interesting Cayley graph is the so-called *Pancake Graph* P_n that arises from connecting permutations π_1 and π_2 if and only if there is a prefix reversal r_i such that $r_i\pi_1 = \pi_2$, with $1 < i \leq n$. Recall that a prefix-reversal is a permutation that in one-line-notation can be written as $r_i = i \ (i-1) \ \cdots \ 1 \ (i+1) \ \cdots \ n$. In other words, r_i reverses the prefix $1 \ 2 \ 3 \ \cdots \ i$ and leaves any element $j > i$ fixed. For example, in S_5 , $r_3 = 32145$ and $r_5 = 54321$. The spectrum of P_n has been studied previously. For instance, Dalfó and Fiol [3] showed that the spectrum of P_n contains all elements from the set $[n-1] \cup \{-1, 0\}$.

Let us use $\mathbb{Z}_m$ to refer to the group of integers modulo m . We refer to the group $\mathbb{Z}_m \wr S_n$, where $\wr$ denotes the standard wreath product, as the group of *colored permutations* (referred to as index permutations by Steingrímsson in [4]). One can write the elements of $\mathbb{Z}_m \wr S_n$ as strings of the form $a_1^{e_1} a_2^{e_2} \cdots a_n^{e_n}$ with $a_i \in [n]$ and $e_i \in \mathbb{Z}_m$ for $1 \leq i \leq n$.

Consider $\pi_1^{e_1} \pi_2^{e_2} \cdots \pi_n^{e_n} \in \mathbb{Z}_m \wr S_n$ and $1 \leq i \leq n$. Then define

$$r_i(\pi_1^{e_1} \pi_2^{e_2} \cdots \pi_n^{e_n}) = \pi_i^{a_i} \pi_{i-1}^{a_{i-1}} \cdots \pi_1^{a_1} \pi_{i+1}^{e_{i+1}} \cdots \pi_n^{e_n}$$

where $a_j = e_j + 1 \pmod m$. Notice that each r_i has an inverse given by

$$r_i^{-1}(\pi_1^{e_1} \pi_2^{e_2} \cdots \pi_n^{e_n}) = \pi_i^{b_i} \pi_{i-1}^{b_{i-1}} \cdots \pi_1^{b_1} \pi_{i+1}^{e_{i+1}} \cdots \pi_n^{e_n}$$

where $b_i = e_i - 1 \pmod m$.

We furthermore refer to the set $R_n := \{r_i, r_i^{-1} : 1 \leq i \leq n\}$ as the set of *generalized prefix reversals*, or *prefix reversals* for simplicity.

A natural variation of pancake graphs is to use the colored permutations $\mathbb{Z}_m \wr S_n$ as set of vertices and the set $\{(s\pi_1, \pi_2) \mid s \in R_n \text{ and } \pi_1, \pi_2 \in \mathbb{Z}_m \wr S_n\}$. This graph is called the *prefix-reversal graphs*, which we denote by $P(m, n)$.

The case $m = 2$ is known as the *Burnt Pancake Graph*, denoted by BP_n . In [1], Blanco and Buehrle proved that the spectrum of BP_n contains all integers in the interval $[0, n] \setminus \{\lfloor n/2 \rfloor\}$.

In this work, the authors extend their result to all prefix reversal graphs with $m > 2$. More precisely, they establish the following theorem.

Theorem 1. *The spectrum of $P(m, n)$ contains all even integers in the set $[0, 2n] \setminus \{2\lfloor n/2 \rfloor\}$.*

Furthermore, one can also consider the *directed prefix-reversal graph* $DP(m, n)$; that is, the graph with vertex set $\mathbb{Z}_m \wr S_n$ and edge set $\{(r_i\pi, \pi) \mid 1 \leq i \leq n, \pi \in \mathbb{Z}_m \wr S_n\}$. Notice that $DP(m, n)$ is obtained by not considering the inverses r_i^{-1} with $1 \leq i \leq n$. We prove that the spectrum of DP_n contains all integers from $[0, n] \setminus \{2\lfloor n/2 \rfloor\}$.

In other words, the spectra of all these prefix-reversal graphs contain a range of consecutive integers or even integers, respectively. This is not a common spectral property of a family of graphs.

In the case where $m = 4k$ for some positive integer k , we can prove that the entire range of even integers in $[0, 2n]$ is included in the spectrum of $P(m, n)$ and that all integers in the interval $[0, n]$ are included in the spectrum of $DP(m, n)$.

One of the notions of the *spectral gap* of a graph G refers to the difference between the largest and second-largest eigenvalue of $A(G)$. Our results provide a bound for the spectral gap of these prefix reversal graphs, though the true spectral gap remains unknown still. The matter of finding the spectral gap for some of these prefix-reversal graphs was also addressed by Chung and Tobin in [2]. Although they did not provide a bound, they observed that the spectral gap seems to be getting smaller as $n \rightarrow \infty$. We are able to observe the same phenomenon in the spectra of $P(m, n)$ and $DP(m, n)$.

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BETWEEN WEAK AND BRUHAT: THE MIDDLE ORDER ON PERMUTATIONS

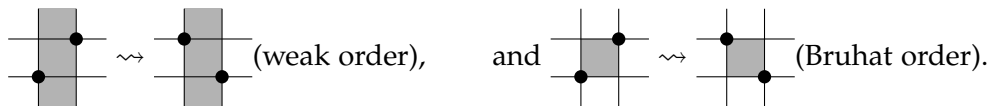
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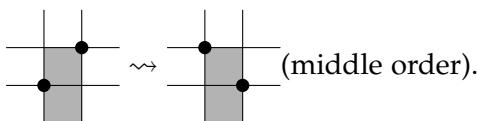
This talk is based on joint work with Luca Ferrari and Bridget E. Tenner

In this contribution, we are interested in partial orders that can be defined on the sets of permutations of any fixed size. Among the most classical such orders are the Bruhat order and the weak order, which motivate the context of our study. (We refer to [1] for background on the Bruhat and weak orders, and to [2] for basics on posets in general.)

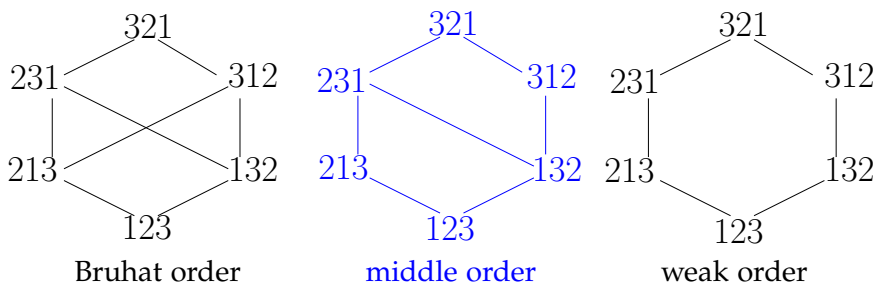
The Bruhat and weak orders are defined on the set S_n of all permutations of size n , and can be conveniently described in terms of the associated covering relations. More specifically, given $\sigma, \tau \in S_n$, the element σ is covered by τ in the Bruhat order when τ is obtained from σ by turning a noninversion into an inversion, provided that no further inversions are created. On the other hand, σ is covered by τ in the weak order when τ is obtained from σ by turning an ascent into a descent. These covering relations can therefore be summarized by the following transformations on occurrences of mesh patterns:



We introduce a class of “intermediate” partial orders, which we call *middle orders*, that interpolate (with respect to refinement) between the Bruhat orders and the weak orders. Their covering relations correspond to the following transformations on occurrences of mesh patterns:



By definition, the middle order refines the weak order on permutations and admits the Bruhat order as a refinement, justifying the terminology. This is illustrated in size 3 by the Hasse diagrams below.



We demonstrate that there is a second combinatorial interpretation of this middle order, in terms of inversion sequences associated with permutations. In this context (and while symmetries are considered in other works), we define the inversion sequence of a permutation σ of size n as $I(\sigma) = (x_1, \dots, x_n)$, where

$$x_i = \#\{j < i \mid \sigma^{-1}(j) > \sigma^{-1}(i)\}.$$

In other words, the i th coordinate of $I(\sigma)$ counts the times that the value i is an inversion top.

We show that σ is less than or equal to τ in the middle order if and only if $I(\sigma)$ is less than or equal to $I(\tau)$, coordinate-wise. As a consequence, as a poset, the middle order is isomorphic to the product of chains $[0, 0] \times [0, 1] \times [0, 2] \times \dots \times [0, n-1]$. Building from there, we can prove nice structural properties of the middle orders: they are distributive lattices, and they are graded.

We proceed by studying the intervals, and the Boolean intervals, of the middle order. We show that the number of intervals (resp. Boolean intervals) in the middle order for size n is $\frac{n!(n+1)!}{2^n}$ (resp. $(2n-1)!!$). Furthermore, denoting $f(n, k)$ (resp. $b(n, k)$) the number of intervals (resp. Boolean intervals) of rank k in the middle order for size n , we show that $f(1, 0) = 1$ and for $n \geq 2$ and $k \in [0, \binom{n}{2}]$,

$$f(n, k) = \sum_{h=0}^{n-1} (n-h) \cdot f(n-1, k-h),$$

(with the convention that $f(n, j) = 0$ when $j < 0$), while we have a non-recursive formula in the Boolean case. Namely, denoting $c(n, j)$ the signless Stirling numbers of the first kind (counting permutations of size n having j cycles, or equivalently permutations of size n having j RtoL-minima), it holds that $b(n, k) = \sum_{i=0}^n \binom{i}{k} c(n, n-i)$ (here, k can only range from 0 to $n-1$).

With the good understanding of the Boolean intervals, and the property of being a distributive lattice, we have easy access to the Möbius function on the middle order (whereas the Möbius function is notoriously difficult to compute in general).

We are also able to give a combinatorial description of the Euler characteristic on the middle order, which counts the number of RtoL-non-minima of a permutation.

This talk and short abstract are based on our article [3].

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This talk is based on joint work with Michal Opler.

The main result we will present in this talk is the following:

Theorem 1. *Every finitely based permutation class whose upper growth rate is less than $1 + \phi \approx 2.618$ is enumerated by a rational generating function.*

It was, in fact, conjectured at Permutation Patterns 2023 that the above theorem should hold up to growth rate 4. This conjecture is still open, and several elements of the methodology in this paper do not extend beyond $1 + \phi = (3 + \sqrt{5})/2$.

There are several steps to proving Theorem 1, the first of which is to prove a conjecture due to Vatter [3] concerning the concentration of cell classes below growth rate $1 + \phi$. A permutation class $\mathcal{C}$ is said to be *concentrated* if there exist integers q and r such that for any $\pi \in \mathcal{C}$, and for any horizontal or vertical line that slices the plot of π , there exists a collection of at most q axis-parallel rectangles in this plot such that: (1) No rectangle is sliced by the line. (2) The rectangles are pairwise *independent*. That is, for any two rectangles their projections onto both the horizontal and vertical axes are disjoint. (3) All but at most r points of π are contained in the rectangles.

For any positive real number γ , the *cell class* $\mathcal{G}_\gamma$ is defined by

$$\mathcal{G}_\gamma = \{ \pi : \text{gr}(\mathcal{S}_\pi^\oplus) < \gamma \text{ or } \text{gr}(\mathcal{S}_\pi^\ominus) < \gamma \}.$$

where $\mathcal{S}_\pi^\oplus$ and $\mathcal{S}_\pi^\ominus$ denote the smallest sum-closed and skew-sum-closed permutation classes containing π , respectively.

Theorem 2. *The cell class $\mathcal{G}_\gamma$ is concentrated if and only if $\gamma < 1 + \phi$.*

The rest of the proof of Theorem 1 extends the grid class methodology of Vatter [2, 3], and then applies an extension of the insertion encoding (see Albert, Linton and Ruškuc [1]) to the resulting structure.

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Hamaker and Rhoades [2] define a *partial permutation* (A, B) of size k as an ordered pair of lists $A = (a_1, \dots, a_k)$ and $B = (b_1, \dots, b_k)$ of positive integers, representing a map $\pi : A \rightarrow B$ such that $\pi(a_i) = b_i$ for all $i = 1, \dots, k$. Pairs (A, B) that represent the same map π are considered to be the same partial permutation.

For any $k \geq m$, a k -*extension* of a partial permutation (A, B) of size j is a permutation $\sigma \in S_k$ such that $\sigma(a_i) = b_i$ for each $i = 1, \dots, m$. In other words, $\sigma|_A = \pi$. A k -*completion* $(A, B)_k$ of a partial permutation (A, B) is the set of all k -extensions of (A, B) .

We call (A, B) and (C, D) k -*Wilf-equivalent as partial permutations* if $(A, B)_k \sim (C, D)_k$ (where “ $\sim$ ” denotes Wilf-equivalence). Furthermore, we call (A, B) and (C, D) *Wilf-equivalent as partial permutations* if $(A, B)_k \sim (C, D)_k$ for all $k \geq \max(A \cup B \cup C \cup D)$. We denote this by $(A, B) \sim (C, D)$.

We prove some and conjecture other Wilf-equivalences of partial permutations, and extend some of those to *shape-Wilf-equivalences* of partial permutations. Many of those can be restated as (shape-)Wilf-equivalences of partially ordered permutations (POPs), proved or conjectured in Burstein et al. [1]. One of those conjectures was recently proved by Wang and Yan [3].

We also establish some enumerative results. For example:

Theorem 1. *We have $((t, t+1), (1, 2))_k \sim (12, 12)_k$ for all $1 \leq t \leq k-1$, and therefore $((t, t+1), (1, 2))$, $t \geq 1$, are Wilf-equivalent as partial permutations. Moreover, for $k \geq 3$,*

$$|\text{Av}_n(((t, t+1), (1, 2))_k)| = \begin{cases} n!, & \text{if } n < k-3, \\ (k-3)! r_{k-3}(n), & \text{if } n \geq k-3, \end{cases}$$

where $r_{k-3}(n)$ is the n -th $(k-3)$ -Schröder number, i.e. the number of Schröder paths with level steps of $k-3$ colors.

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ON INVERSIONS, MAJOR INDEX, AND THE AREA STATISTIC ON ℓ -INTERVAL PARKING FUNCTIONS

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This talk is based on joint work with Jennifer Elder, Kimberly P. Hadaway, Pamela E. Harris, Jeremy L. Martin, Amanda Priestley, and Gabe Udell

Consider the following parking scenario. There are n cars attempting to park in n spots on a one way street, and their preferred parking spot of each car is recorded as a tuple $\alpha = (a_1, a_2, \dots, a_n) \in [n]^n$. As each car parks, it drives to its preferred spot and parks there if it is available. If that spot is not available, the car continues driving and parks in the next available spot. If all the cars are able to park in their preferred spot, or at most ℓ away from that spot, we call α an ℓ -interval parking function. Let $\text{IPF}_n(\ell)$ denote the class of ℓ -interval parking functions of length n . These were introduced by Aguilar-Fraga et al. [1] generalizing work of Colaric et al. in [6].

Enumerative results on ℓ -interval parking functions with $\ell \geq 1$ also give connections between this set of parking functions and Dyck paths with restricted height and to preferential arrangements [1]. However, one aspect that is not well understood is the discrete statistics of these combinatorial objects. In this talk, we study inversion, major index and area statistics on the set of ℓ -interval parking functions.

We start with a natural question: Are inversion and major index equidistributed on the set of ℓ -interval parking functions? Clearly, the answer is “yes” for $\ell = 0, n - 1$ since these are permutations and parking functions, respectively. But what about the ℓ in between? We provide a complete answer: first, recall the classical *Foata transform* on words turns a word w into a word w' such that $\text{inv}(w) = \text{maj}(w')$.

Theorem 1. *Let $1 \leq \ell \leq n - 2$.*

1. *If $\ell \in \{1, 2, n - 2\}$, then the Foata transform restricts to a bijection $\text{IPF}_n(\ell) \rightarrow \text{IPF}_n(\ell)$.*
2. *If $\ell \notin \{1, 2, n - 2\}$, then inversion and major index are not equidistributed on $\text{IPF}_n(\ell)$.*

The $\ell = 1$ in fact follows from the existence of $\mathcal{S}_n$ -action on $\text{IPF}_n(1)$ [5]. Interestingly, there does not appear to be any $\mathcal{S}_n$ -action for $\ell = 2, n - 2$.

Next, we use area to give enumerative formulas for ℓ -interval parking functions. For a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$ and index $i \in [n]$, define

$$L_\ell(i; \sigma) = \min(\ell + 1, i - t + 1) \tag{1}$$

where $\sigma_t, \sigma_{t+1}, \dots, \sigma_i$ is the longest contiguous subsequence of σ such that $\sigma_k \leq \sigma_i$ for all $k \in [t, i]$. The case $\ell = n - 1$ (which includes all parking functions, without the ℓ -interval restriction) was studied in [7]. Recall that the *area* of a parking function $\alpha \in \text{PF}_n$ is the quantity $\text{area}(\alpha) = \sum_{i=1}^n i - \alpha_i$.

Proposition 2 ([7] $\ell = n - 1$). *Let $0 \leq \ell \leq n - 1$. For all $\sigma \in \mathcal{S}_n$, we have*

$$\sum_{\alpha \in \text{IPF}_n(\ell)} q^{\text{area}(\alpha)} = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^n [L_\ell(i; \sigma)]_q.$$

We call a 1-interval parking function a *unit interval parking function* and denote $\text{UPF}_n = \text{IPF}_n(1)$. This set is enumerated by the Fubini numbers [4, 10] and unit interval parking functions has been shown to be connected to many other famous combinatorial objects such as the combinatorial game of the Tower of Hanoi [2], the facets of the permutahedron [5], and Boolean intervals in the weak Bruhat order of the symmetric group [9] to name a few. We can refine our results for unit interval parking functions by considering area and inversions simultaneously.

Corollary 3. *For all $n \geq 1$, $\sum_{\alpha \in \text{UPF}_n} q^{\text{area}(\alpha)} t^{\text{inv}(\alpha)} = \sum_{\sigma \in \mathcal{S}_n} (1 + q)^{\text{asc}(\sigma)} t^{\text{inv}(\sigma)}$.*

Let X be a finite set, let $X(q)$ be a polynomial in q such that $X(1) = |X|$, let $C = \langle g \rangle$ be a cyclic group of order n acting on X , and let ω be a primitive complex n th root of unity. The triple $(X, X(q), C)$ is said to exhibit the *cyclic sieving phenomenon* (CSP) [11] if $X(\omega^j) = |\{x \in X : c^j(x) = x\}|$ for every $c \in C$.

As shown in [5, Lemma 3.12, Proposition 3.13, Corollary 3.14], there is an $\mathcal{S}_n$ -action on UPF_n that fixes area. The cyclic subgroup C_n of $\mathcal{S}_n$ generated by the cycle $g = (1, 2, \dots, n) \in \mathcal{S}_n$ acts on UPF_n by restricting the $\mathcal{S}_n$ -action on UPF_n to C_n . For $k \in \mathbb{N}$, let $\text{UPF}_{n,k}$ be the set of unit interval parking functions with area k and set

$$f_{n,k}(t) = \sum_{\alpha \in \text{UPF}_{n,k}} t^{\text{inv}(\alpha)}.$$

Theorem 4. *The triple $(\text{UPF}_{n,k}, f_{n,k}(t), C_n)$ exhibits CSP for each n and k .*

Finally, the *Lehmer code* of a permutation σ is the sequence $L(\sigma) = (a_1, \dots, a_n)$ so that $a_i = \#\{j < i \mid \sigma^{-1}(i) > \sigma^{-1}(j)\}$. The map $\sigma \mapsto L(\sigma)$ defines a bijection between $\mathcal{S}_n$ and the set $E_n := \prod_{i=0}^{n-1} [0, i]$. Unit interval parking functions also have a notion of Lehmer code: a *cipher* is a pair (w, I) of $w \in E_n$ and a selection of positions $I = \{i_1, \dots, i_k\}$ such that $\text{Asc}(w) \subseteq I$, viewing w as a word. Let $\mathcal{G}_n$ denote the set of ciphers of length n . The following theorem is a consequence of work by Avalos and Bly [3].

Theorem 5. *There is a bijection $\psi : \text{UPF}_n \rightarrow \mathcal{G}_n$ such that for each $\alpha \in \text{UPF}_n$, if $\psi(\alpha) = (w, I)$, then $\text{inv}(\alpha) = \sum_{i=1}^n w_i$, $|I| = \text{area}(\alpha)$ and if σ_i is the car parking in spot i , then $L(\sigma_1 \cdots \sigma_n) = w$.*

Ciphers are a great way of understanding unit interval parking functions. For instance, they can very quickly allow you to derive a new proof of the following.

Corollary 6 ([9, Theorem 1.2]). *Unit interval park functions with j inversions and $n - k$ blocks all of size 1 or 2 are in bijection with rank k boolean intervals $[u, v]$ of $\mathcal{S}_n$ under the right weak order with $\text{inv}(u) = j$.*

To conclude, we discuss a number of different statistics that we would like to know more about for ℓ -interval parking functions.

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DESCENTS OF PERMUTATIONS WITH ONLY EVEN OR ONLY ODD CYCLES

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Let $\mathcal{S}_n^o$ denote the set of permutations in $\mathcal{S}_n$ all of whose cycles have odd length. Let $\mathcal{S}_n^e$ be the set of permutations all of whose cycles have even length, except possibly for one cycle of length one (i.e., a fixed point). It is known that $|\mathcal{S}_n^o| = |\mathcal{S}_n^e|$ for all n , as one can show using exponential generating functions. When n is even, a bijective proof appears in Bóna's book [2, Lem. 6.20], and it is not hard to extend it to odd n .

In a recent preprint [1], Adin, Hegedűs and Roichman proved that the identity $|\mathcal{S}_n^o| = |\mathcal{S}_n^e|$ has a surprising refinement. For $\pi \in \mathcal{S}_n$, denote its ascent and descent sets by $\text{Asc}(\pi) = \{i \in [n-1] : \pi_i < \pi_{i+1}\}$ and $\text{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$.

Theorem 1 ([1]). *For any n and any subset $J \subseteq [n-1]$,*

$$|\{\pi \in \mathcal{S}_n^o : \text{Asc}(\pi) = J\}| = |\{\pi \in \mathcal{S}_n^e : \text{Des}(\pi) = J\}|.$$

The proof in [1] relies on a new generating function identity on higher Lie characters, so a natural question is whether Theorem 1 has a bijective proof. Unfortunately, Bóna's bijection between $\mathcal{S}_n^o$ and $\mathcal{S}_n^e$ does not behave well with respect to descent sets.

Our main result is a bijective proof of Theorem 1. More specifically, we prove the following, from where Theorem 1 follows by the principle of inclusion-exclusion.

Theorem 2. *For any n and any subset $S \subseteq [n-1]$, there exists an explicit bijection*

$$f : \{\pi \in \mathcal{S}_n^o : \text{Asc}(\pi) \subseteq S\} \rightarrow \{\pi \in \mathcal{S}_n^e : \text{Des}(\pi) \subseteq S\}. \quad (2)$$

Our proof combines two known bijections between permutations and multisets of necklaces, together with a new bijection for Lyndon factorizations of words. We summarize the main ideas below. For more details and examples, we refer the reader to [3].

Let $S = \{s_1, s_2, \dots, s_{k-1}\}_< \subseteq [n-1]$. Denote its associated composition by $\alpha = \alpha(S) = (s_1, s_2 - s_1, \dots, s_{k-1} - s_{k-2}, n - s_{k-1})$, and define the monomial $\mathbf{x}^\alpha = \prod_{i=1}^k x_i^{\alpha_i}$. Fix a totally ordered alphabet $A = \{a_1, a_2, \dots, a_k\}$. Denote by $\mathcal{W} = A^*$ the set of finite words over A , and by $\mathcal{W}_n$ the set of those of length n . Two words $u, v \in \mathcal{W}$ are *conjugate* if they are cyclic rotations of each other. A *necklace* is a conjugacy class of words in $\mathcal{W}$. A word u is *primitive* if it is not the power of another word. A necklace is *primitive* if it is the conjugacy class of a primitive word.

Let $\mathcal{M}_n$ be the set consisting of all multisets of primitive necklaces of total length n . The *cycle structure* of $M \in \mathcal{M}_n$ is the partition of n whose parts are the lengths of the necklaces in the multiset, and its *weight* is the monomial $\text{wt}(M) = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$, where α_i is the number of times that a_i appears in M . Define the *weight* of a word $w \in \mathcal{W}$ similarly. Let $\mathcal{M}_n^o$ be the set of elements in $\mathcal{M}_n$ which consist of *distinct* necklaces of

odd length, and let $\mathcal{M}_n^e$ be the set of those which consist of necklaces of *even* length, except possibly for one necklace of length one.

We start by applying a bijection due to Gessel and Reutenauer [5, Lem. 3.4] to the right-hand side of equation (2). Even though the original bijection is defined on words, we interpret it as a map from permutations to multisets of necklaces that preserves the cycle structure. When applied to $\mathcal{S}_n^e$, we get the following.

Proposition 3. *There exists a bijection*

$$\Phi_S : \{\pi \in \mathcal{S}_n^e : \text{Des}(\pi) \subseteq S\} \rightarrow \{M \in \mathcal{M}_n^e : \text{wt}(M) = \mathbf{x}^{\alpha(S)}\}.$$

The bijection Φ_S does not work well with permutations whose ascent set is contained in S . Instead, to deal with the left-hand side of equation (2), we use a different bijection that, in a slightly different form, has appeared in work of Gessel, Restivo and Reutenauer [4, Sec. 3], and is also a special case of a bijection due to Steinhardt [7]. In general, the necklaces in the image of this bijection may not be primitive or distinct, but they are when we restrict to $\mathcal{S}_n^o$.

Proposition 4. *There exists a bijection*

$$\Xi_S : \{\pi \in \mathcal{S}_n^o : \text{Asc}(\pi) \subseteq S\} \rightarrow \{M \in \mathcal{M}_n^o : \text{wt}(M) = \mathbf{x}^{\alpha(S)}\}.$$

Consider the lexicographic order on $\mathcal{W}$, which we denote by $<$. A primitive word in $\mathcal{W}$ is called a *Lyndon word* if it is strictly smaller than all the other words in its conjugacy class; equivalently, if it is strictly smaller than all of its proper suffixes [6, Prop. 5.1.2]. Lyndon words are in one-to-one correspondence with primitive necklaces, since each conjugacy class of primitive words has a unique lexicographically smallest element. Denote by $\mathcal{L}$ the set of Lyndon words in $\mathcal{W}$. The following is a well-known result of Lyndon.

Theorem 5 ([6, Thm. 5.1.5]). *Every $w \in \mathcal{W}$ has a unique Lyndon factorization, that is, an expression $w = \ell_1 \ell_2 \dots \ell_m$ where $\ell_i \in \mathcal{L}$ for all i , and $\ell_1 \geq \ell_2 \geq \dots \geq \ell_m$.*

Identifying primitive necklaces with Lyndon words allows us to view multisets of primitive necklaces as Lyndon factorizations of words: each necklace in $M \in \mathcal{M}_n$ becomes a Lyndon factor of the associated word $w \in \mathcal{W}_n$.

Let $\mathcal{W}_n^o$ be the set of words in $\mathcal{W}_n$ all of whose Lyndon factors have *odd* length and are *distinct*. Let $\mathcal{W}_n^e$ be the set of words in $\mathcal{W}_n$ all of whose Lyndon factors have *even* length, except possibly for one factor which has length one. The above identification gives straightforward bijections between $\mathcal{M}_n^o$ and $\mathcal{W}_n^o$, and between $\mathcal{M}_n^e$ and $\mathcal{W}_n^e$. Using Propositions 3 and 4, Theorem 2 is now equivalent to the following.

Theorem 6. *There exists a weight-preserving bijection $\Psi : \mathcal{W}_n^o \rightarrow \mathcal{W}_n^e$.*

To describe the map Ψ , we need the one more definition. The *standard factorization* of a Lyndon word $w \in \mathcal{L} \setminus A$ is the expression $w = rs$ where s is the longest proper suffix

of w that belongs to $\mathcal{L}$, or equivalently, the lexicographically smallest proper suffix of w . It is known [6, Prop. 5.1.3] that, in this case, $r \in \mathcal{L}$ and $r < rs < s$.

Given $w \in \mathcal{W}_n^o$, we build $\Psi(w)$ by repeatedly applying certain updates to a pair of words (O, E) . Initially, $(O, E) = (w, -)$, where $-$ denotes the empty word. Each step moves some subword from O to the beginning of E . At any time, all the Lyndon factors of O are odd and distinct, and all the Lyndon factors of E are even. At the end of the algorithm, we have $(O, E) = (-, \Psi(w))$.

Definition 7 (The bijection Ψ). On input $w \in \mathcal{W}_n^o$, initially set $(O, E) = (w, -)$, and iterate the following step as long as $|O| \geq 2$:

- Let $O = o_1 o_2 \dots o_m$ be the Lyndon factorization of O . Say that o_m is *splittable* if $|o_m| \geq 2$ and its standard factorization $o_m = rs$ satisfies $s < o_{m-1}$ (if $m = 1$, then o_m is splittable by convention). Update (O, E) to

$$(O', E') = \begin{cases} (o_1 o_2 \dots o_{m-1} r, sE) & \text{if } o_m \text{ is splittable and } r \text{ has odd length,} \\ (o_1 o_2 \dots o_{m-1} s, rE) & \text{if } o_m \text{ is splittable and } r \text{ has even length,} \\ (o_1 o_2 \dots o_{m-2}, o_m o_{m-1} E) & \text{if } o_m \text{ is not splittable.} \end{cases}$$

If we reach $|O| = 1$ (this case only occurs when n is odd), move this letter to the Lyndon factorization of E by inserting it as a new factor, in the unique location that keeps the factors weakly decreasing from left to right.

Once O is empty, let $\Psi(w) = E$.

With some work, we can show that Ψ is a weight-preserving bijection $\Psi : \mathcal{W}_n^o \rightarrow \mathcal{W}_n^e$, and therefore the composition $f = \Phi_S^{-1} \circ \Psi \circ \Xi_S$ proves Theorem 2.

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PATTERNS IN MEANDRIC SYSTEMS AND TREE-INDEXED SUMS OF CATALAN NUMBERS

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This talk is based on joint work with Alin Bostan and Paul Thévenin

We study *meandric systems* of size n , which are collections of non-crossing loops intersecting the horizontal axis exactly at the points $0, \dots, 2n - 1$ (up to continuous deformation fixing the horizontal axis). Combinatorially, meandric systems can be uniquely represented as a pair (P_1, P_2) of non-crossing pair-partitions; in particular there are Cat_n^2 meandric systems of size n , where Cat_n is the n -th Catalan number.

Given a number i in $\{1, \dots, 2n\}$ and a meander system $M = (P_1, P_2)$, we are interested in the "pattern" or "shape" (of the component) of i , defined as follows. We see M as multigraph, where vertices are the integers from 0 to $2n - 1$ and the edges are given by the pairs in P_1 and P_2 . In this construction, each vertex has degree 2 and the connected components are closed loops. Let us write $C_i(M)$ for the component of i , and $2k$ for its number of vertices. Then the pattern $\text{Pat}_i(M)$ of i in M is obtained by relabelling the vertices of $C_i(M)$ with the unique increasing bijection $V(C_i(M)) \rightarrow \{0, \dots, 2k - 1\}$. An example is given on Fig. 9. By construction, $\text{Pat}_i(M)$ is always a meandric system with a single connected component, aka a meander.

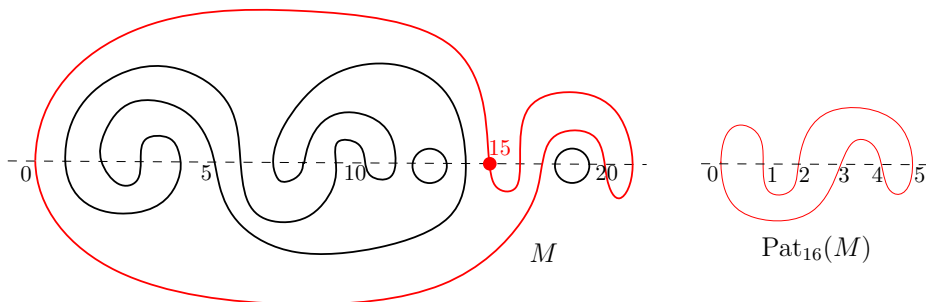


Figure 9: Left: a meandric system M . Its vertices of the meandric system are labelled from 0 to 21, but to make the picture more readable, we only typed every fifth label. We look at the pattern of the red vertex $i = 15$ in M . The component of i is drawn in red; it contains the vertices 0, 16, 17, 18, 20 and 21. To get the pattern $\text{Pat}_i(M)$ (shown on the right part of the picture), we relabel them as 0, 1, 2, 3, 4 and 5 respectively.

Given a fixed meander M_0 , we are interested in the large n limit of the probability $\mathbb{P}(C_i(M_n) = M_0)$, where M_n is a uniform random meandric system of size n , and i is a uniform random integer in $\{0, \dots, 2n - 1\}$, independent from M_n . We call this

quantity the *asymptotic frequency* of M_0 . For example, we can prove that

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(C_i(M_n) = \bigcirc \right) = \frac{1}{8} \sum_{\ell=0}^{\infty} \text{Cat}_{\ell}^2 2^{-4\ell} = \frac{2}{\pi} - \frac{1}{2} \approx 0.137 \quad (3)$$

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{P} \left(C_i(M_n) = \text{---} \right) &= \frac{1}{64} \cdot \sum_{\ell_1, \ell_2, \ell_3 \geq 0} \text{Cat}_{\ell_1} \text{Cat}_{\ell_2} \text{Cat}_{\ell_3} \text{Cat}_{\ell_1+\ell_3} 2^{-4\ell_1-2\ell_2-4\ell_3} \\ &= \frac{1}{4} - \frac{2}{3\pi} \approx 0.038. \end{aligned} \quad (4)$$

Our first result, proved in [1] and generalizing the first equality in (3), (4), is a combinatorial summatory expression for the asymptotic frequency of M_0 . To state it, we need to introduce some notation. First, for a meander M_0 , we consider its dual trees as shown on Fig. 10: their vertices are the bounded regions delimited by the meander and the horizontal axis, and two vertices are connected, if the corresponding regions share an interval of the horizontal axis as boundary. We also add an half-edge to a vertex whenever the corresponding region is connected to an unbounded region.

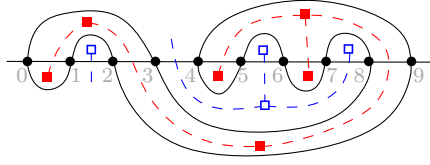


Figure 10: A meander and its dual trees.

Second, given a tree T , possibly with an half-edge, we consider the sum

$$S(T) := \sum_{(x_e) \in \mathbb{Z}_+^{E(T)}} \left(\prod_{v \in V(T)} \text{Cat}_{X_v} 4^{-X_v} \right), \quad (5)$$

where for a vertex v , we set $X_v = \sum_{e \ni v} x_e$ (i.e. we sum x_e over all edges incident to v). For example, if P_2 (resp. P_2^h) is the path with two vertices linked by an edge, (resp. with an additional half-edge), we have by definition:

$$\begin{aligned} S(P_2) &= \sum_{x \in \mathbb{Z}_+} (\text{Cat}_x 4^{-x})^2 \\ S(P_2^h) &= \sum_{x_1, x_2 \in \mathbb{Z}_+} \text{Cat}_{x_1} 4^{-x_1} \text{Cat}_{x_1+x_2} 4^{-x_1-x_2}. \end{aligned}$$

Proposition 1. *For any meander M_0 of size k , we have*

$$\lim_{n \rightarrow +\infty} \mathbb{P}(C_i(M_n) = M_0) = 2^{-4k+1} k S(T_1) \cdots S(T_m) \quad (6)$$

where $T_1, \dots, T_m$ are the dual trees of M_0 .

Our second result, proved in [2] and generalizing the second equality in (3), (4), is a structural result on these numbers.

Theorem 2. *For any tree T with zero or one half-edge, the multiple sum $S(T)$ is a polynomial in $1/\pi$. Consequently, the asymptotic frequency $\lim_{n \rightarrow +\infty} \mathbb{P}(C_i(M_n) = M_0)$ of any meander M_0 is a polynomial in $1/\pi$.*

The proof is constructive and uses both the linear and the quadratic recurrences of Catalan numbers, together with properties of the hypergeometric functions ${}_2F_1$.

We also get a compact expression for stars. Namely, if T_s is the tree with a central vertex connected with s leaves, then

$$S(T_{s+3}) = \frac{64}{\pi} \cdot \left(\sum_{k=0}^s \binom{s}{k} \cdot \frac{1}{(2k+1)(2k+3)(2k+5)} \right).$$

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This talk is based on joint work with Eli Bagno, Toufik Mansour, and Amir Safadi

In the last decades, the area of pattern avoidance on set partitions of type A has received a lot of attention (see [11] and references therein). In particular, Sagan [15] offered two different approaches to pattern-avoidance in set partitions of type A , one of them, following Klazar [8, 9], was based on avoiding some structures of blocks of the set partitions (where the blocks are ordered increasingly with respect to their minimal elements), while the other used their restricted-growth (RG-)words representation. Sagan [15, Prop. 4.1] proved that pattern-avoidance in the set partitions sense implies pattern-avoidance in the RG-words representation sense, but not the other way around.

It seems that the first interest in pattern-avoidance in set partitions of type B arose in the seminal paper of Reiner [14], who introduced the notion of set partitions of type B in such a way that it corresponds to the plane arrangement of the Coxeter group of type B . Athanasiadis [1] calculated the number of noncrossing and nonnesting partitions of fixed block sizes for types B and C and for most of the partitions of type D . His work was completed by Athanasiadis and Reiner in [2].

Most of the results presented in this research were obtained using the method of *generating trees*, presented briefly below: Any set $\mathcal{C}$ of discrete objects with a notion of a size such that for each n , there are finitely many objects of this size is called a *combinatorial class*. West [16] introduced the notion of a *generating tree* for $\mathcal{C}$, which is a rooted, labeled tree whose vertices are the objects of $\mathcal{C}$ with the following properties:

1. Each object of $\mathcal{C}$ appears exactly once in the tree.
2. Objects of size n appear at level n in the tree (the root has level 0).
3. The children of every object are obtained by a set of succession rules in a form that determines the number of children and their labels.

This idea has been further utilized in closely related problems, see e.g. in [4, 5, 6, 7]. This technique has been used to count pattern-avoiding classes of some combinatorial objects, see for example [10, 12].

In this research, we make use of this method to count pattern-avoidance classes of RG-words of type B . We introduced the requested definitions:

Definition 1 ([3]). Let Σ^B be the alphabet $\{0, \pm 1, \pm 2, \dots, \pm n\}$ and define the following order on Σ^B : $0 \prec -1 \prec 1 \prec -2 \prec 2 \prec \dots \prec -n \prec n$.

A *restricted-growth (RG-)word of type B of the second kind of length n* is a word $\omega = \omega_1 \cdots \omega_n$ in the alphabet Σ^B which satisfies the following conditions:

- (1) We have $\omega_1 = 0$ or $\omega_1 = 1$.
- (2) For each $2 \leq t \leq n$, the following inequality holds:

$$\omega_t \preceq \max \{|\omega_1|, \dots, |\omega_{t-1}|\} + 1,$$

with respect to the order defined above.

In the case that

$$|\omega_t| = \max \{|\omega_1|, \dots, |\omega_{t-1}|\} + 1, \quad (7)$$

we demand: $\omega_t > 0$.

Denote by $\mathcal{R}^B(n, k)$ the set of RG-words of type B of length n whose maximal element is k . Moreover, define $\mathcal{R}_n^B = \bigcup_k \mathcal{R}^B(n, k)$ and $\mathcal{R}^B = \bigcup_{n \geq 0} \mathcal{R}_n^B$.

Let $\omega = \omega_1 \cdots \omega_n \in \mathcal{R}_n^B$ and let $\tau = \tau_1 \cdots \tau_k$ for some $1 \leq k \leq n$ be any word over the alphabet Σ^B such that $\{|\tau_1|, \dots, |\tau_k|\} = \{0, 1, \dots, d\}$ for some $d \leq n$. We say that ω contains τ , if there is a sequence of k indices, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\omega_{i_a} X \omega_{i_b}$ if and only if $\tau_a X \tau_b$, for all $1 \leq i < j \leq k$ and $X \in \{<, =, >\}$. In such a context, τ is usually called a *signed pattern*. For example, the RG-word of type B : $001(-1)(-1)21$ avoids 210 but contains 100 (due to the subsequence $1(-1)(-1)$). We denote the set of all τ -avoiding RG-words of signed set partitions in $\mathcal{R}_n^B$ by $\mathcal{R}_n^B(\tau)$. For an arbitrary finite collection of patterns P , we say that ω avoids P if ω avoids each $\tau \in P$; we denote the corresponding subset of $\mathcal{R}_n^B$ by $\mathcal{R}_n^B(P)$. We say that two sets of patterns P and P' are Wilf-equivalent if $|\mathcal{R}_n^B(P)| = |\mathcal{R}_n^B(P')|$, for all $n \geq 0$. Let P be any set of nonempty patterns. Define $\mathcal{R}^B(P) = \bigcup_{n=0}^{\infty} \mathcal{R}_n^B(P)$.

Inspired by Mansour and Safadi [12], we construct a generating tree $\mathcal{T}(P)$ for the class of pattern-avoiding RG-words of signed set partitions $\mathcal{R}^B(P)$. Starting with the root ϵ which occupies level 0, we recursively construct additional nodes of the tree $\mathcal{T}(P)$ such that the n th level of the tree consists of exactly the elements of $\mathcal{R}_n^B(P)$ arranged so that the parent of an RG-word $\omega_1 \cdots \omega_n \in \mathcal{R}_n^B(P)$ is the unique RG-word $\omega_1 \cdots \omega_{n-1} \in \mathcal{R}_{n-1}^B(P)$. The children of $\omega_1 \cdots \omega_{n-1} \in \mathcal{R}_{n-1}^B(P)$ are all the elements $\omega_1 \cdots \omega_{n-1} \omega_n$ from the set of the RG-words of type B of length n , satisfying the pattern-avoiding restrictions of the patterns in P . Figure 11 presents the first few levels of $\mathcal{T}(\{01\})$. Clearly, the cardinality of $\mathcal{R}_n^B(P)$ is the number of nodes in the n th level of $\mathcal{T}(P)$.

The main achievement of this research is a classification of pattern-avoidance for a single pattern of length 2 and 3.

For patterns of length 2, we have the following result:

Theorem 2. For all $n \geq 0$,

$$(1) \sum_{n \geq 0} |\mathcal{R}_n^B(00)| \frac{x^n}{n!} = (1+x)e^{x+\frac{x^2}{2}},$$

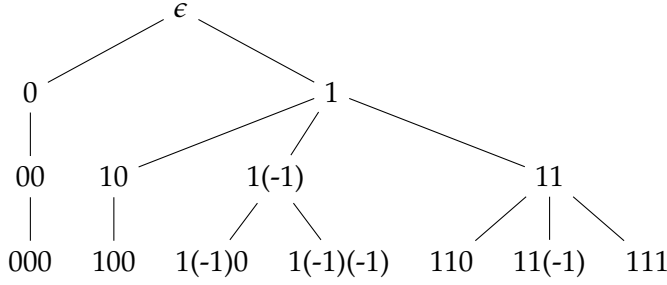


Figure 11: The generating tree $\mathcal{T}(\{01\})$

$$(2) \sum_{n \geq 0} |\mathcal{R}_n^B(01)| x^n = \frac{1-x+x^2}{(1-x)^3},$$

$$(3) \sum_{n \geq 0} |\mathcal{R}_n^B(10)| x^n = \frac{1}{1-2x},$$

For patterns of length 3, Table 1 presents the first terms of the sequence $\{|\mathcal{R}_n^B(\tau)|\}_{n=1}^9$, for all such patterns τ ordered by τ ; none of them appears in the OEIS [13]. We denote $o\checkmark$ ($e\checkmark$, resp.) where we have computed the ordinary (exponential, resp.) generating function. It is noteworthy that some of the proofs use combinatorial arguments.

τ	$\{ \mathcal{R}_n^B(\tau) \}_{n=1}^9$	Enumeration
000	2, 6, 22, 98, 486, 2692, 16346, 107382, 756748	
001, 010	2, 6, 22, 87, 357, 1517, 6677, 30407, 143027	
011	2, 6, 22, 87, 361, 1554, 6907, 31609, 148664	$e\checkmark$
012	2, 6, 22, 84, 315, 1138, 3941, 13093, 41857	$o\checkmark$
021	2, 6, 22, 86, 339, 1322, 5069, 19084, 70583	$o\checkmark$
100	2, 6, 22, 92, 432, 2224, 12392, 74064, 470944	$e\checkmark$
101, 110	2, 6, 22, 91, 412, 2002, 10306, 55709, 314146	
102, 120	2, 6, 22, 85, 330, 1276, 4916, 18901, 72602	$o\checkmark$
201, 210	2, 6, 23, 100, 467, 2285, 11559, 59960, 317201	$o\checkmark$

Table 1: The sequence $\{|\mathcal{R}_n^B(\tau)|\}_{n=1}^9$, for all patterns τ of length three.

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This talk is based on joint work with Oscar Lopez and Mike Weiner

Using generating trees, West [1] showed that the class $\mathcal{S}(2413, 3142)$ of separable permutations is counted by the large Schröder numbers. In this talk, we will discuss how to enumerate these permutations by the absolute position of the 1 (in one-line notation), and by the position of the 1 relative to the position of the maximal entry. We will show that, as suspected by West, enumeration by the position of the 1 leads to an alternative way to arrive at the Schröder numbers.

Let

$$S(x) = 1 + x + 2x^2 + 6x^3 + 22x^4 + 90x^5 + 394x^6 + \dots$$

be the generating function for the sequence $a_0 = 1$, $a_n = |\mathcal{S}_n(2413, 3142)|$, and let $\mathcal{S}_n^{\ell \mapsto 1}$ be the subset of elements in $\mathcal{S}_n(2413, 3142)$ having the 1 at position ℓ . That is,

$$\mathcal{S}_n^{\ell \mapsto 1} = \{\sigma \in \mathcal{S}_n(2413, 3142) : \sigma(\ell) = 1\}.$$

Proposition 1. *The generating function $g(x, u) = \sum_{n=1}^{\infty} \sum_{\ell=1}^n |\mathcal{S}_n^{\ell \mapsto 1}| u^\ell x^n$ satisfies*

$$g(x, u) = \frac{xuS(x)S(xu)}{S(x) + S(xu) - S(x)S(xu)}.$$

Proof. Let $g_i(x, u)$ and $g_d(x, u)$ be the components of $g(x, u)$ counting the corresponding indecomposable and decomposable permutations of size greater than 1. Thus,

$$\begin{aligned} g_i(x, u) &= u^2x^2 + (u^2 + 2u^3)x^3 + \dots, \\ g_d(x, u) &= ux^2 + (2u + u^2)x^3 + \dots, \end{aligned}$$

and $g(x, u) = xu + g_i(x, u) + g_d(x, u)$. Note that, since every permutation is either indecomposable or has an indecomposable factor, we have

$$g(x, u) = (xu + g_i(x, u))S(x) \quad \text{and} \quad g_d(x, u) = (xu + g_i(x, u))(S(x) - 1). \quad (8)$$

Since the reverse map is an involution on $\mathcal{S}_n(2413, 3142)$, we have

$$g(x, u) = ug(xu, \frac{1}{u}) = (xu + ug_i(xu, \frac{1}{u}))S(xu).$$

On the other hand, since the reverse of an indecomposable permutation is decomposable, we also have $g_d(x, u) = ug_i(xu, \frac{1}{u})$, and therefore

$$\begin{aligned} g(x, u) &= (xu + g_d(x, u))S(xu) \\ &= xuS(x)S(xu) + g_i(x, u)(S(x) - 1)S(xu). \end{aligned}$$

Combining this with (8), we arrive at the equation

$$(xu + g_i(x, u))S(x) = xuS(x)S(xu) + g_i(x, u)(S(x) - 1)S(xu),$$

which gives

$$g_i(x, u) = \frac{xuS(x)(S(xu) - 1)}{S(x) + S(xu) - S(x)S(xu)}.$$

Using again (8), we then get

$$g(x, u) = (xu + g_i(x, u))S(x) = \frac{xuS(x)S(xu)}{S(x) + S(xu) - S(x)S(xu)}.$$

□

Remark 2. Letting $u = 1$, Proposition 1 gives the equation

$$g(x, 1) = S(x) - 1 = \frac{xS(x)S(x)}{S(x) + S(x) - S(x)S(x)} = \frac{xS(x)}{2 - S(x)},$$

which leads to $S(x) = \frac{1}{2}(3 - x - \sqrt{1 - 6x + x^2})$.

Enumeration by relative position

For $a, k \geq 1$, let $\mathcal{S}_{n,k}^{a \prec n}(2413, 3142)$ be the set of $\sigma \in \mathcal{S}_n(2413, 3142)$ such that:

- $\sigma^{-1}(n) - \sigma^{-1}(a) = k$,
- $\sigma^{-1}(b) - \sigma^{-1}(n) > 0$ for every $b \in \{1, \dots, a-1\}$.

For $a > 1$, any permutation $\sigma \in \mathcal{S}_{n,k}^{a \prec n}(2413, 3142)$ must be of the form $\sigma = \pi \ominus \tau$, where $\pi \in \mathcal{S}_{m,k}^{1 \prec m}(2413, 3142)$ with $m = n - a + 1$, and $\tau \in \mathcal{S}_{a-1}(2413, 3142)$. For this reason, it is enough to just examine the case $a = 1$.

Theorem 3. *The generating function $f(x, t) = \sum_{n,k \geq 1} |\mathcal{S}_{n,k}^{1 \prec n}(2413, 3142)| t^k x^n$ satisfies*

$$f(x, t) = \frac{x^2 t S(x) S(xt)^2}{(S(xt) + S(x) - S(xt)S(x))^2}.$$

As a consequence, $\sum_{n,k,a \geq 1} |\mathcal{S}_{n,k}^{a \prec n}(2413, 3142)| t^k s^a x^n = f(x, t) \cdot sS(xs)$.

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Let π_n be a uniformly chosen random permutation on $[n]$. The authors of Allen et al. [1] showed that the expected number of distinct *consecutive* patterns of all lengths $k \in \{1, 2, \dots, n\}$ in π_n was $\frac{n^2}{2}(1 - o(1))$ as $n \rightarrow \infty$, exhibiting the fact that random permutations pack *consecutive* patterns near-perfectly. A conjecture was made in a 2018 thesis of Fokuoh [3] that the same is true for *non-consecutive* patterns, i.e., that there are $2^n(1 - o(1))$ distinct non-consecutive patterns expected in a random permutation. In this talk we will indicate a proof of this conjecture.

A key combinatorial lemma, of interest in its own right, gives a bound on the probability that $2k - r$ positions, consisting of two k windows with an overlap of r , contain isomorphic patterns. It is of key value in the proof of the conjecture; the rest is “just analysis”.

This work is joint with Yonah Biers-Ariel, Jane Street Capital, Verónica Borrás-Serrano, University of Puerto Rico at Mayagüez, Isabel Byrne, University of Delaware, Nathaniel Veimau, Microsoft Azure, and Yiguang Zhang, Columbia University and Google.

The second topic is related to permutations through matrix decompositions. It was presented in Moscow at the Problem Session. The Rose McCarty Conjecture [2] is that any McCarty Matrix (an $n \times n$ matrix A with positive integer entries and each of the $2n$ row and column sums equal to n), can be additively decomposed into two other matrices, B and C , such that B has row and column sumsets both equal to a permutation π on $[n]$, and C has row and column sumsets both equal to a permutation of $\{0, 1, \dots, n-1\}$. The problem can also be formulated in terms of bipartite graphs.

The McCarty conjecture was formulated by Rose McCarty in 2015. In fact, the main focus of the work of Eastham, Kay, McCarty, and Spencer [2] was to reduce the upper bound on the “total acquisition number” of a diameter two graph, from $32 \ln n \ln \ln n$ (LeSaulnier et al. [4]) to a substantially smaller number. They proved that, if the McCarty conjecture was true, then this upper bound could be reduced to 4. Since we prove the McCarty conjecture in this paper, we are able to report that the total acquisition number a_t of a diameter two graph is indeed at most 4. The alternative graph theoretic formulation of the McCarty Conjecture is as follows.

McCarty’s Conjecture, the bipartite multigraph version: If M is a loopless bipartite multigraph on $2n$ vertices with bipartitions X and Y such that $|X| = |Y| = n$ and for each $v \in V(B)$, $\deg(v) = n$. Suppose furthermore that there exists a subgraph H of B with the same vertex set such that for every integer $k \in [n]$, there exist $x \in X, y \in Y$ such that $\deg_H(x) = \deg_H(y) = k$. Then for all diameter 2 graphs G , $a_t(G) \leq 4$.

We used probabilistic methods to solve the conjecture. This work is joint with Lybitina Koene, Virginia Tech, and Grant Shirley, ETSU.

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ONE-WAY BOUNDED PERMUTATIONS AND THE FLATORIAL NUMBERS

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This talk is based on joint work with Brett Stevens and Aaron Williams

We consider *one-way bounds* on a permutation $p_1 p_2 \cdots p_n$ for a fixed parameter $r \geq 0$. For example, the *low descent* bound is $p_i > p_{i+1}$ implies $p_{i+1} \leq r$ (i.e., the small value in a descent is at most r). The *short right* bound is $p_i^{-1} - i \leq r$ (i.e., values move at most r positions to the right relative to $12 \cdots n$). No bounds are given in the opposite direction (i.e., on ascents or leftward moves). We show that eight types of permutations are counted by the *flatorial number* $\langle n, r \rangle!$ which is $n!$ for $n < r$ and $r!(r+1)^{n-r}$ for $n \geq r$.

Introduction

Let S_n be the set of all permutations of $[n] = \{1, 2, \dots, n\}$ in one-line notation. Consider the following two subsets of S_n for $n = 4$ and a fixed parameter $r = 1$ explained below.

$$\text{rgt}(4, 1) = \{\underline{1}234, 2\underline{1}34, 23\underline{1}4, 32\underline{1}4, 234\underline{1}, 243\underline{1}, 324\underline{1}, 423\underline{1}\} \quad (9)$$

$$\begin{aligned} \overline{\text{inv}}(4, 1) &= \{1234, 1243, 1324, 1342, \underline{2}134, \underline{2}143, \underline{2}314, \underline{2}341\} \\ &\quad 0000 \quad 0001 \quad 0010 \quad 0011 \quad 0100 \quad 0101 \quad 0110 \quad 0111 \end{aligned} \quad (10)$$

The first set contains permutations $p_1 p_2 \cdots p_n$ in which $p_i^{-1} > i$ implies $i \leq r$. In other words, if a value i appears further to the right than it does in the identity, then $i \leq r$. Since $r = 1$ in (9), only value 1 (underlined) can appear to the right of its position in id. The second set contains permutations whose inversion vector³ (shown below) $v_1 v_2 \cdots v_n$ has $v_i \leq r$ for all i . In other words, every value is inverted with at most r smaller values. Since $r = 1$ in (10), each value is inverted with at most one smaller value.

The two sets have the same cardinality; we will prove that this is true for all n and r . More broadly, we show that eight different types of permutations are counted by the *flatorial number* $\langle n, r \rangle!$. These numbers are like factorial numbers except that terms in the product are at most $r + 1$. In particular, $\langle n, 0 \rangle! = 1$ and $\langle n, 1 \rangle! = 2^{n-1}$ counts (9)–(10).

$$\langle n, r \rangle! = 1 \cdot 2 \cdots r \cdot (r+1)^{n-r} = \frac{n!}{(n)_{n-r}} = \begin{cases} n! & \text{if } n < r \\ r! \cdot (r+1)^{n-r} & \text{if } n \geq r \end{cases}. \quad (11)$$

None of our equivalences are obtained by reversing indices $p_n \cdots p_2 p_1$ and/or inverting values $(n - p_1 + 1)(n - p_2 + 1) \cdots (n - p_n + 1)$. These trivial modifications known as the *symmetries of the square* extend our results to $8 \cdot 4 = 32$ equinumerous types. For

³The *inversion vector* counts the smaller inverted values. That is, $v_i = |\{j \mid j < i \text{ and } p_j^{-1} > p_i^{-1}\}|$.

example, the “low rights” in (9) become “high lefts” (i.e., $p_i^{-1} < i$ implies $i \geq n - r + 1$) by reversing and inverting. Our results are summarized in Table 2. While each type is fairly elementary, we were unable to find many references in the literature. Table 3 shows that only three of the types are listed in corresponding OEIS entries. Short descents with $r = 2$ arise using classic pattern avoidance [?]: $\overline{\text{dsc}}(n, 2) = \text{Av}_n(312, 231)$. Some non-classical avoidance results also appear in the OEIS sequences [?, ?].

low descents	short descents	low rights	short rights	large invert	small invert	early lefts	short lefts
insert value	insert value	swap value	swap value	insert index	insert index	swap index	swap index
small labels	large labels	small labels	large labels	small labels	large labels	small labels	large labels
61584327	18564732	78516243	53814276	1783642	13584726	75483216	15768243
$\text{dsc}(n, r)$	$\overline{\text{dsc}}(n, r)$	$\text{rgt}(n, r)$	$\overline{\text{rgt}}(n, r)$	$\text{inv}(n, r)$	$\overline{\text{inv}}(n, r)$	$\text{lft}(n, r)$	$\overline{\text{lft}}(n, r)$

Table 2: Eight types of one-way bounded permutations (see Definition 2). Each is counted by the flatorial $\langle n, r \rangle!$ via a family subtree with large ($\ell - r$ to ℓ) or small (ℓ and 1 to r) branch labels. The sample permutations satisfy their bound for $r = 4$ (or larger) but not $r = 3$. For example, $p = 61584327 \in \text{dsc}(8, 4) \setminus \text{dsc}(8, 3)$ as the smaller value in each descent is ≤ 4 but not ≤ 3 .

r	Formula ($n \geq r$)	$n = 1, 2, 3, \dots$	OEIS	short rights	short ascent	short lefts
1	$1! \cdot 2^{n-1}$	1, 2, 4, 8, 16, 32, 64, ...	A000079	Arndt (2009)		
2	$2! \cdot 3^{n-2}$	1, 2, 6, 18, 54, 162, 486, ...	A025192		Lewis (2006)	
3	$3! \cdot 4^{n-3}$	1, 2, 6, 24, 96, 384, 1536, ...	A084509		Knuth (2022)	
4	$4! \cdot 5^{n-4}$	1, 2, 6, 24, 120, 600, 3000, ...	A179364 [†]			Hardin (2010)
5	$5! \cdot 6^{n-5}$	1, 2, 6, 24, 120, 720, 4320, ...	A179365 [†]			Hardin (2010)

Table 3: Flatorial numbers for small r . Three of our permutation types appear in the OEIS entries. [†]Several sequences match the initial terms and have another constraint (e.g., [A179357](#)).

Family Trees of Permutations & One-Way Bounded Permutations

We consider four *family trees* of permutations. Each tree has root 1 and S_ℓ appears at level ℓ . At level ℓ , the branches are labeled with values $b \in [\ell]$. The branch ℓ always inserts ℓ as the rightmost symbol, and identity permutations are on rightmost paths. The remaining children are obtained as follows, where $b \in [\ell - 1]$ is the branch label.

- *Insert value tree*: insert ℓ to the left of value b .
- *Swap value tree*: insert ℓ as the rightmost symbol, then swap it with the value b .
- *Insert index tree*: insert ℓ to the left of the index b .
- *Swap index tree*: insert ℓ as the rightmost symbol, then swap it with index b .

For each family tree we consider two subtrees parameterized by a given value of $r \geq 0$.

- The *large label subtree* includes branches with labels $\ell - r$ to ℓ at level ℓ .
- The *small label subtree* includes branches with labels ℓ and 1 to r at level ℓ .

So at level ℓ the subtrees include branches with label ℓ and (at most) the r smallest or largest remaining labels. Thus, the number of nodes at each successive level increases from 1 up to $r + 1$ giving Remark 1. Family trees and subtrees appear in Figures 12–13.

Remark 1. A family subtree with large (or small) labels has $\langle \ell, r \rangle!$ nodes at level ℓ .

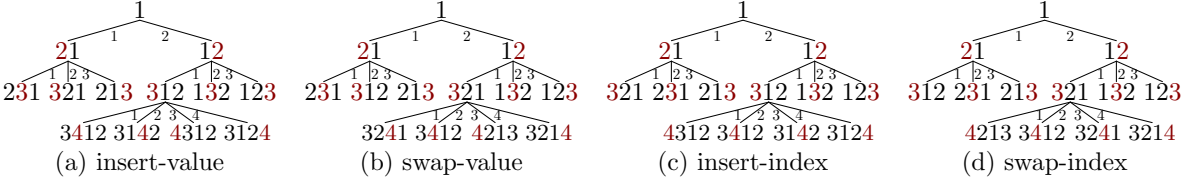


Figure 12: Four family trees up to level $\ell = 3$ with one node at level $\ell = 4$. A branch with label $b < \ell$ inserts ℓ (a) before value b or (c) at index b , or inserts ℓ in the last position and then swaps it with (b) value b or (d) index b . A branch with label $b = \ell$ inserts ℓ into the last position.

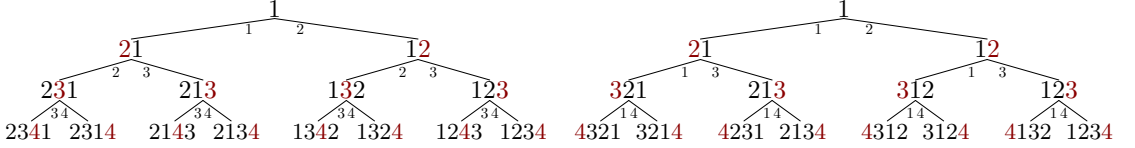


Figure 13: The insert index family subtree with large (left) and small (right) labels for $r = 1$. Note that the ℓ branches are included in both subtrees.

Definition 2. Let $p = p_1 p_2 \cdots p_n$ be a permutation, $p^{-1} = p_1^{-1} p_2^{-1} \cdots p_n^{-1}$ its inverse, and $v = v_1 v_2 \cdots v_n$ its inversion vector. Value $i \in [n]$ is an *invert* if it is inverted with some smaller value (i.e., $v_i > 0$). For example, $p = 2413$ has $p^{-1} = 3142$, $v = 0102$, and inverts 2 and 4. We define the following *one-way bounded permutations* for a fixed $r \geq 1$.

1. p has *low descents*, denoted $p \in \underline{\text{dsc}}(n, r)$, if $p_{i+1} < p_i$ implies $p_{i+1} \leq r$. That is, the smaller value in a descent is at most r .
2. p has *short descents*, denoted $p \in \overline{\text{dsc}}(n, r)$, if $p_i - p_{i+1} \leq r$. That is, each descent has height at most r .
3. p has *low rights*, denoted $p \in \underline{\text{rgt}}(n, r)$, if $p_i^{-1} > i$ implies $i \leq r$. That is, only small values can move to the right.
4. p has *short rights*, denoted $p \in \overline{\text{rgt}}(n, r)$, if $p_i^{-1} - i \leq r$, or equivalently, $p_i \geq i - r$. That is, values can move at most r spaces to the right.
5. p has *large inverts*, denoted $p \in \underline{\text{inv}}(n, r)$, if $v_i > 0$ implies $v_i \geq i - r$. That is, any invert is inverted with at least $i - r$ smaller values.
6. p has *small inverts*, denoted $p \in \overline{\text{inv}}(n, r)$, if $v_i \leq r$ for all i . That is, each value is inverted with at most r smaller values.
7. p has *early lefts*, denoted $p \in \underline{\text{lft}}(n, r)$, if $p_i^{-1} < i$ implies $p_i^{-1} \leq r$. That is, if a value is moved to the left, it must be in the first r positions of p .
8. p has *short lefts*, denoted $p \in \overline{\text{lft}}(n, r)$, if $p_i^{-1} \geq i - r$, or equivalently, $p_i \leq i + r$. That is, values can move at most r spaces to the left.

Theorem 3. For all $n \geq 1$ and $r \geq 1$, each of the sets of one-way bounded permutations in Definition 2 are counted by flatorial numbers, $\langle n, r \rangle!$.

Proof. By Remark 1, for each set of one-way bounded permutations, we need only show that the permutations in the set are exactly the nodes in the associated subtree with large labels or small labels. We illustrate this for low rights using the swap value tree with small labels; the other proofs are similar. A *right* is a value v with $p_v^{-1} > v$.

Consider the swap value tree, in which each child of $p_1 p_2 \dots p_{\ell-1}$ at branch b is obtained by appending ℓ , and then swapping ℓ with b for $b < \ell$. Each branch $b < \ell$ creates a

right: b is now in position $\ell > b$. Suppose $p = p_1 p_2 \dots p_{\ell-1} \in \underline{\text{rgt}}(\ell-1, r)$. The children of p in $\underline{\text{rgt}}(\ell-1, r)$ are exactly those with branches $b \leq r$ (in which rights with values at most r are created), and branch ℓ (in which no right is created). On the other hand, if v is a right in permutation p , then v will be a right in all descendants of p , since v either remains in its same position for branches $b \neq v$, or moves further to the right at any branch $b = v$. Therefore, if $p \notin \underline{\text{rgt}}(n, k)$, then none of its descendants can be. $\square$

This talk is based on joint work with Sergey Kirgizov

Kitaev [3], along with Mansour [4, 5], presented the enumeration of classes of permutations avoiding at least two length 3 consecutive patterns. In this work, we study the *asymptotic popularity*, that is, the limit probability to find a given pattern of size 3 at random position in a random permutations in the eighteen avoidance classes from Kitaev-Mansour works. We show that, in certain cases, some of the remaining patterns disappear asymptotically. It is a quite enchanting fact. The first known occurrence of this "emerging consecutive pattern avoidance" is found in [1], where the authors proved that the consecutive pattern 321 asymptotically disappears in the class $\text{Av}(123, 132, 213)$.

Table 4 summarizes our results, presenting eighteen classes, as they appear in [3]. In this table, an empty cell means that corresponding patterns are avoided *by design*, while 0 says that the respective pattern disappears asymptotically (the probability to find this pattern at a random position in a random permutation tends to 0, as the permutation size grows). The values 1/2 and 1/4 present in this table should also be understood in the asymptotic sense. There are two "N/A" for Class 4, because this class is empty for $n > 3$. Question marks indicate open problems.

For most of the classes, the study of the asymptotic popularities is quite straightforward from the structure of the permutations in the class. However, we focused our attention on two classes where this problem is significantly harder.

Avoiding 123, 132 and 321

The first class we study in details is Class 11, namely $\text{Av}(123, 132, 321)$. To do so, we use an enumerative approach, computing the exact number of occurrences of each pattern in the class.

Theorem 1. For $p \in \{213, 231, 312\}$ and $n \in \mathbb{N}$, let $\mathbf{p}_n$ denote the total number of occurrences of $\mathbf{p}$ in the permutations of $\text{Av}_n(123, 132, 321)$. Then,

$$\begin{aligned} 231_n &= (n-1)!! \left\lceil \frac{n-3}{2} \right\rceil + (n-2)!! \left\lceil \frac{n-2}{2} \right\rceil, \\ 312_n &= (n-1)!! \left(\frac{(-1)^{n-1} + n - 3}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ k \neq n \pmod{2}}}^{n-1} \frac{1}{k} \right) \\ &\quad + (n-2)!! \left(\frac{(-1)^n + n - 4}{4} + \frac{1}{2} \sum_{\substack{k=1 \\ k \equiv n \pmod{2}}}^{n-2} \frac{1}{k} \right), \end{aligned}$$

Class \ Pattern	123	132	213	231	312	321
1 (simple)			1/2	1/2		
2 (simple)				0		1
3 (simple)	1/2					1/2
4 (simple)			N/A		N/A	
5 (simple)	1			0		
6 (simple)				1/2	1/2	
7 (done in [1])				1/2	1/2	0
8 (simple)			0		0	1
9 (simple)	1/2				0	1/2
10 (open)			?	?		?
11 (Theorem 1)			1/4	1/2	1/4	
12 (open)		?	?			?
13 (open)		?	?		?	?
14 (open)	?	?			?	?
15 (open)	?			?	?	?
16 (simple)		1/4	1/4	1/4	1/4	
17 (Theorem 2)			1/4	1/2	1/4	0
18 (simple)	1/2		0		0	1/2

Table 4: The asymptotic popularity patterns among eighteen avoidance classes.

$$213_n = (n-1)!! + (n-2)!! - 231_n - 312_n,$$

where $n!!$ is defined by $0!! = 1$, and for $n \geq 1$

$$n!! = \begin{cases} n \cdot (n-2) \dots 3 \cdot 1 & \text{if } n \text{ is odd,} \\ n \cdot (n-2) \dots 4 \cdot 2 & \text{if } n \text{ is even.} \end{cases}$$

The asymptotic popularities of 231, 312 and 213 can then be deduced directly from Theorem 1 (see Table 4).

Avoiding 123 and 132

The other class we focused our attention on is Class 17, namely $\text{Av}(123, 132)$. In [2], Claesson proved that the Foata transform induces a bijection between $\text{Av}_n(123, 132)$ and $\mathcal{I}_n$, the set of involutions of size n . We then take advantage of the rarity of fixed points in $\mathcal{I}_n$ to compute the asymptotic popularity of 321 and 231 (see Table 4). Table 5 shows the correspondence between patterns in Class 17 and patterns in $\mathcal{I}_n$.

For 213 and 312, thanks to the Foata transform, we show that it is enough to study the popularity of the pattern 2314, which we do analytically.

Theorem 2. Let $G(z) = \sum_{n=4}^{+\infty} \frac{2314_n}{n!} z^n$ be the EGF of $(2314_n)_{n \geq 4}$. Then

$$G(z) = \frac{e^{\frac{(1+z)^2}{2}}}{2} \int_0^z e^{-\frac{(1+t)^2}{2}} dt + \frac{z(z-2)e^{z+\frac{z^2}{2}}}{4}.$$

Pattern in $\text{Av}_n(123, 132)$	Pattern in $\mathcal{I}_n$, with $a < b < c$	Fixed point-free pattern in $\mathcal{I}_n$
321	$(c)(b)(a)$ or $(c)(b)(a \star)$ or $(\star c)(b)(a)$	$\emptyset$
231	$(b c)(a)$ or $(b c)(a \star)$	$(b c)(a \star)$
213	$(b)(a c)$ or $(\star b)(a c)$	$(\star b)(a c)$
312	$(c)(a b)$ or $(\star c)(a b)$	$(\star c)(a b)$

Table 5: The correspondence between patterns in $\text{Av}_n(123, 132)$ and $\mathcal{I}_n$.

With Theorem 2 and some complex analysis, we can then deduce the asymptotic popularity of 213 and 312 (see Table 4).

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This talk is based on joint work with Michal Opler

Let M be a $r \times c$ matrix with entries $\{-1, 0, 1\}$. A *monotone grid class* $\text{Grid}(M)$ is a set of all permutations that admit a grid-like decomposition into $r \times c$ blocks such that the block in the i -th column and j -th row induces an increasing permutation if $M_{i,j} = 1$, a decreasing one if $M_{i,j} = -1$ and the block is empty if $M_{i,j} = 0$. The class $\text{Grid}(M)$ is *acyclic* if there is no cycle in M , i.e., no sequence of its distinct non-zero entries $c_0, c_1, \dots, c_{k-1}$ such that the c_i and $c_{(i+1) \bmod k}$ are either in the same column or in the same row and the whole sequence is neither in a single column nor in a single row.

Table 6: Selected results

Class	M	Generating function
L_1	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$-\frac{2x^4 - 6x^3 + 4x^2 - x}{2x^4 - 9x^3 + 12x^2 - 6x + 1}$
L_4	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$-\frac{x^4 - 4x^3 + 3x^2 - x}{(1-x)^2(1-3x+x^2)}$
T_5	$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	$\frac{(14x^5 - 56x^4 + 71x^3 - 39x^2 + 10x - 1)x}{(2x^2 - 4x + 1)(x^2 - 3x + 1)(3x - 1)(x - 1)^2}$
J_3	$\begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$-\frac{(3x^4 - 13x^3 + 17x^2 - 7x + 1)x}{(2x^2 - 4x + 1)(x^2 - 3x + 1)(2x - 1)}$
J_{10}	$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$	$-\frac{(2x^4 - 9x^3 + 13x^2 - 6x + 1)x}{(2x^2 - 4x + 1)(2x - 1)(x - 1)^2}$
Z_3	$\begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$-\frac{(3x^5 - 4x^4 - 15x^3 + 18x^2 - 7x + 1)x}{(x^2 - 3x + 1)(3x - 1)(2x - 1)(x - 1)}$
W_{10}	$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$	$-\frac{(x^5 - 7x^4 + 19x^3 - 18x^2 + 7x - 1)x}{(x^3 - 6x^2 + 5x - 1)(x^2 - 3x + 1)(x - 1)}$

Braunfeld [2] recently showed how to represent acyclic monotone grid classes⁴ in monadic second order logic (MSOL). We implement a Python program which given an acyclic matrix M generates the proposed representation of $\text{Grid}(M)$ in MONA language. MONA [3] is then used to generate a deterministic finite automaton A over alphabet with 2^k symbols where k is the number of non-zero entries in M (but the automaton uses only k letters of this alphabet).

An important property of this representation is that there exists a bijection between permutations in $\text{Grid}(M)$ and strings accepted by A , and moreover this bijection

⁴The paper talks about geometric grid classes but we do not support general geometric grid classes yet, only acyclic ones.

is length preserving. Hence to count how many permutations of size n belong in $\text{Grid}(M)$, it is enough to count strings of length n accepted by A . We use Sage [4] to obtain generating function counting these string by solving system of linear equations obtained from transition matrix of A .

Moreover, our implementation admits imposing additional restrictions on permutations in the class like being simple, sum- or skew-indecomposable or avoiding given pattern.

We use this program to calculate generating functions and bases of some acyclic monotone grid classes. We are able to do this for most classes with 4 and some with 5 non-zero entries obtaining new results and confirming some conjectured ones [1]. Some of them with simple generating functions are shown in Table 6. We also intend to make this program publicly available.

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THE WELL QUASI-ORDER PROBLEM FOR COMBINATORIAL STRUCTURES UNDER THE CONSECUTIVE ORDER

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This talk is based on joint work with Nik Ruškuc

A partially ordered set is *well quasi-ordered* (wqo) if it contains neither infinite descending sequences nor infinite antichains. This property provides a notion of ‘tameness’ for partially ordered sets, with those which are not wqo being comparatively ‘wild’. Well quasi-order has been widely studied across combinatorics, with major theorems including Higman’s Lemma [2], Kruskal’s Tree Theorem [6] and the Graph Minor Theorem [4].

It can be seen that every subset of a wqo set is also wqo. On the other hand, non-wqo sets may have subsets which are wqo or non-wqo, and we will ask when these subsets are wqo? In particular, we will consider well quasi-order for subsets of partially ordered sets, called *avoidance sets*, which are defined by their forbidden substructures. Given a partially ordered set $(C, \leq)$ and $B \subseteq C$, the avoidance set of B is the set:

$$\text{Av}(B) = \{x \in C : b \not\leq x \ \forall b \in B\}.$$

Avoidance sets give rise to natural algorithmic decidability questions: given a finite set B , we ask about decidability of properties of $\text{Av}(B)$. We will focus on one such question: the *well quasi-order problem*. This asks if it is decidable, given a finite set B , whether $\text{Av}(B)$ is wqo? In other words, is there an algorithmic distinction between wqo and non-wqo avoidance sets [1]?

The objects of study for this talk will be partially ordered sets (posets) of combinatorial structures in which two elements are related when one is a substructure of the other, in some sense. For example, permutations under the classical and consecutive subpermutation orders are two such posets. While the wqo problem is notably open under the classical order, progress has been made for the consecutive order, under which the wqo problem has been shown to be decidable [3]. To attain this result, McDevitt and Ruškuc encoded permutations as paths in certain factor graphs, and found conditions on factor graphs which determine wqo for avoidance sets of permutations.

This talk will explore recent generalisations of these techniques, which enable us to tackle the wqo problem for a range of structures under consecutive orders. To extend the notion of the consecutive order to combinatorial structures more generally, we relate two structures when one embeds in the other such that ‘consecutive’ elements remain consecutive in the image. We will begin by defining factor graphs for such posets and demonstrating how we can often encode structures as paths in these factor graphs.

We will explore the solution to the wqo problem in the cases where each path encodes more than one structure, and show how this yields decidability of the wqo

problem for a wide class of structures, including graphs, digraphs, and collections of relations. Similarly, we will see that the wqo problem is decidable when each path encodes exactly one structure. These two cases provide insight into conditions forbidding wqo for the intermediate cases where paths may be associated with one or more structures, and invite investigation into these cases. We will highlight the solution to the wqo problem for two interesting examples of structures for which this occurs: permutations and equivalence relations [3], [5]. Additionally, we will discuss connections between the results presented so far and compare the various conditions on factor graphs determining wqo, showing that perhaps a spectrum of conditions is emerging.

To finish, we will present some examples of posets of structures in which intriguingly we cannot encode elements as paths in factor graphs in the way described so far. We will outline how we may solve the wqo problem for these examples using variations of the techniques discussed during the talk.

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This talk is based on joint work with Robert Brignall

The existence of the infinite antichain of oscillations at growth rate $\kappa \approx 2.20557$ marks an important phase transition in the set of possible growth rates of permutation classes. This is the smallest infinite antichain, and Vatter [1] and Bevan [2] have used it to construct permutation classes of every growth rate greater than $\lambda_A \approx 2.35698$. We begin by explaining that these constructions would in fact work for *any* infinite antichain of permutations, thus motivating a systematic search for antichains as a means of understanding the finer structure of the set of permutation class growth rates. Motivated by a desire to generalise the antichain at κ using pin sequences (a construction first introduced by Brignall, Huczynska and Vatter [3] as a means of studying simple permutations), we introduce a remarkable permutation class $\mathcal{V}_c$ occurring at growth rate $\nu_c \approx 3.51205$: this is a well-quasi-ordered class (meaning it contains no infinite antichains) whose one-point extension $\mathcal{V}_c^{+1}$ nevertheless contains uncountably many distinct infinite antichains. We shall show that these so-called $\mathcal{V}$ -antichains can be generated using binary sequences over $\{0, 1\}$ and that their growth rates depend only on the *recurrent complexity* of the defining binary sequence (the sequence counting the number of factors that recur infinitely often). This results in extraordinarily large collections of distinct infinite antichains occurring at the same growth rates, and will allow us to identify an interesting new point of phase transition in the set of growth rates of permutation classes, at a constant $\nu_{\mathcal{L}} \approx 3.28277$. We shall show that this constant $\nu_{\mathcal{L}}$ is an accumulation point of genuinely distinct infinite antichains from both above and below, as well as demonstrating the appearance at $\nu_{\mathcal{L}}$ of uncountably many distinct *minimal* antichains. We can also demonstrate the existence of a (very thin) interval occurring close to 3.5 in which every number is the growth rate of an antichain. These results strongly suggest that a full classification of antichains beyond growth rate $\nu_{\mathcal{L}}$ may be impossible, but also motivate a plausible conjecture about the growth rates of antichains below this point.

Finally, taking the relationship between $\mathcal{V}$ -antichains and binary sequences to its conclusion, we consider the study of factorial languages (that is, factor-closed sets of words over $\{0, 1\}$) in its own right and introduce the *realisability problem*: that of determining whether a given factorial language can be realised as the set of recurrent factors of some binary sequence. We shall prove some partial results towards this problem and consider the consequences for a potential further classification of antichains.

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This talk is based on joint work with Priyavrat Deshpande and Anurag Singh

Given a fixed pattern, one can associate a simplicial complex to any permutation by collecting its subsequences that avoid the pattern. Recently, researchers have begun to take interest in understanding the topology of these complexes. In this ongoing work, we investigate the relation between structural properties of the permutation (usually relating to patterns) and topological/combinatorial properties of these complexes.

Preliminaries

Given $\pi \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_k$, with $k \leq n$, we define a simplicial complex X_π^σ on $[n]$ as follows: a subset $\{i_0, \dots, i_k\}$ forms a simplex if the corresponding subsequence $\pi_{i_0} \cdots \pi_{i_k}$ avoids σ . We refer the reader to Kozlov's book [2] for all the relevant terminology concerning simplicial complexes. For example, the faces of X_π^{21} correspond to the occurrences of identity permutations (of various lengths) in π . The aim of our project is to investigate the relationship between the topology of X_π^σ and the combinatorics of permutation patterns; with the hope of bringing a more structural and qualitative aspect to enumerative and discrete problems.

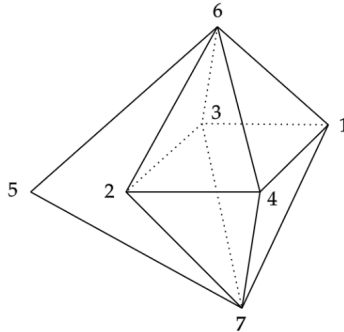


Figure 14: [3, Figure 1] showing X_π for $\pi = 3254176$. Note that $X_\pi \simeq S^1 \vee S^2$.

We will mainly focus on the case $\sigma = 21$ and mention results for other patterns σ in the last section. We denote X_π^{21} by just X_π . Now we briefly state the connection of X_π to other areas of mathematics and some recent work. Recall that the *inversion graph* G_π of $\pi \in \mathfrak{S}_n$ is the graph with vertex set $[n]$ and there is an edge between i and j if $i < j$ and $\pi(i) > \pi(j)$. The so-called *independence complex* of G_π is precisely X_π ; this is because the increasing subsequences in π correspond to independent subsets of G_π .

The first appearance of X_π , in a slightly different guise, can be traced to [4]; where the authors prove that the complex X_π is homotopy equivalent to a (disjoint union of) wedge of spheres. This is a rather surprising result. Such simplicial complexes are

sought after by researchers in topological combinatorics. A reproof of this result has also appeared in [1]. In a recent article, Meshulam-Moyal [3] consider the problem of computing the probability that X_π is not topologically r -connected. This paper is one of the motivations for this project. Since X_π is a wedge of spheres, one should try and find the exact decomposition (not just the connectivity) in terms of properties related to permutation patterns.

Results

A simplicial complex is determined by its facets (maximal faces). Note that the complex X_π is not necessarily pure, *i.e.*, its facets need not have the same dimension. A facet of dimension $k - 1$ corresponds to an occurrence of the mesh pattern $(12 \cdots k, \{(i, i) \mid i \in [0, k]\})$. In particular, facets of dimension 0 in X_π correspond to *strong fixed points* of the reverse of π . A strong fixed point in σ is an index i such that $\sigma(j) < i$ for all $j < i$, and $\sigma(j) > i$ for all $j > i$.

A particular class of patterns plays an important role in the study of X_π . For any $k \geq 0$, the *cross-pattern of dimension k* is given by $\text{cp}_k = \oplus^{k+1} 21$, the direct sum of $(k + 1)$ copies of 21. Note that $X_{\text{cp}_k} \simeq S^k$, the k -sphere, since X_{cp_k} can be viewed as the boundary of the k -dimensional cross-polytope. The importance of these permutations comes from the following result. Recall that the homotopy dimension of a space is the minimum dimension of a cell complex homotopy equivalent to it.

Proposition 1. *For $\pi \in \text{Av}(\text{cp}_k)$, the homotopy dimension of X_π is at most $k - 1$.*

We also characterize the permutations that maximize homotopy dimension.

Proposition 2. *For any $\pi \in \mathfrak{S}_n$, the homotopy dimension of X_π is at most $\lfloor \frac{n}{2} \rfloor - 1$. This upper bound is achieved only by $\text{cp}_{\frac{n}{2}-1}$, if n is even and by precisely $n^2 - 5 \binom{n-1}{2} - 1$ permutations, if n is odd.*

One can also relate the existence of spheres in the homotopy type of X_π to patterns in π . The following result can also be stated using the language of mesh patterns.

Proposition 3. *If a permutation π contains an occurrence of the pattern cp_k such that at least one increasing subsequence in this occurrence corresponds to a facet of X_π , then the complex contains an embedded k -sphere, which is homologically non-trivial.*

We also have interesting results for special classes of permutations.

Proposition 4. *Let $\pi \in \text{Av}_n(3412)$ be an involution and $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$ where each π_i is sum indecomposable.*

1. X_π is disconnected if and only if $\pi(1) = n$ (equivalently, $k = 1$).
2. X_π is contractible if and only if π has a strong fixed point.
3. If $k \geq 2$ and π has no strong fixed points, then X_π is $(k - 2)$ -connected.

4. The complex X_π is pure if and only if each π_i is decreasing.

We have similar results for other special classes of permutations such as Grassmannian permutations and permutations avoiding a pattern of lengths 3. For example, we have the following result.

Proposition 5. *The permutations $\pi \in \text{Av}_n(132)$ such that X_π is pure correspond to Dyck paths of semilength n that have all peaks at the same height, which are counted by [A007059](#).*

Cohen-Macaulay simplicial complexes, which lie at the crossroads of combinatorics, topology, and commutative algebra, are of interest to many researchers. One of the aims of our project is to identify, in terms of pattern information, which of these complexes are Cohen-Macaulay. In this regard, we have the following result.

Proposition 6. *If $\pi \in \text{Av}(2143)$, then the complex X_π is vertex decomposable (strongest of the Cohen-Macaulay conditions).*

Future directions and open problems

1. From the results above, if $\pi \in \text{Av}(2143)$ is skew indecomposable, then X_π is contractible. Similarly, if π has a strong fixed point, then X_π is contractible. Characterize π such that X_π is contractible.
2. Characterize the possible homotopy types of X_π , where π varies over permutations with k descents. We have done so for $k = 0, 1, 2$. We have also shown that, for a fixed k , there are only finitely many possible homotopy types.
3. Characterize π for which X_π is pure. These are the permutations where every increasing subsequence is part of a longest increasing subsequence.
4. Let δ_k denote the decreasing permutation of size k . We now have that for all $k \geq 3$, the complex $X_\pi^{\delta_k}$ has the homotopy type of a wedge of spheres. Investigate other properties of this complex.

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This talk is based on joint work with Nicolas Bonichon and Adrian Tanasa ⁵

Baxter permutations and Baxter d -permutations

Baxter permutations are a central family of permutations that are in relation with numerous other combinatorial objects. They are also known to be characterised by the forbidden vincular patterns $2-41-3$ and $3-14-2$. The *separable permutations*, are also a well studied permutation class that form a subset of the Baxter permutations. A permutation is separable if it is of size 1 or if it is the skew or direct sum of two separable permutations. These permutations are additionally characterised by the forbidden patterns 2413 and 3142.

In this presentation, we are interested in higher dimensional analogs of these permutation classes. In higher dimensions, permutations are generalised by *multidimensional permutations*, also called d -permutations [3, 4]. A d -permutation of $[n]$ is a sequence of $d-1$ permutations $\pi = (\pi_1, \dots, \pi_{d-1})$. Given a d -permutation π , π_0 is the identity permutation on $[n]$. The *diagram* of a d -permutation π is the set of points in $P_\pi := \{(\pi_0(i), \pi_1(i), \dots, \pi_{d-1}(i)), i \in [n]\}$. Figure 16 shows (on the right) an example of 3-permutation.

Definition 1. Let $\mathbf{i} = i_1 \dots i_{d'}$ be a sequence of indices in $\{0, \dots, d\}$, let also $\sigma = (\sigma_1, \dots, \sigma_{d-1})$ be a d -permutation of size n . The *projection* on $\mathbf{i}$ of σ is the d' -permutation given by $\text{proj}_{\mathbf{i}}(\sigma) := (\sigma_{i_2}\sigma_{i_1}^{-1}, \dots, \sigma_{i_{d'}}\sigma_{i_1}^{-1})$. A projection is *direct* if $i_1 < i_2 < \dots < i_{d'}$.

Definition 2. Let the d -permutation $\sigma = (\sigma_1, \dots, \sigma_{d-1}) \in S_n^{d-1}$ and the d' -permutation $\pi = (\pi_1, \dots, \pi_{d'-1}) \in S_k^{d'-1}$ with $k \leq n$ and $d' \leq d$. Then σ *contains the pattern* π if there exists a direct projection $\sigma' = \text{proj}_{\mathbf{i}}(\sigma)$ of dimension d' and a set of indices $c_1 < \dots < c_k$ s.t. $\sigma'_i(c_1) \dots \sigma'_i(c_k)$ is order-isomorphic to π_i for all i . A permutation avoids a pattern if it doesn't contain it. Let s be a symmetry operation of the $[n]^d$ grid (seen as a d -cube), $s(P_\pi)$ is a diagram of a d -permutation that we denote $s(\pi)$. We also denote by $\text{Sym}(\pi)$ the family of permutations obtained by applying the symmetries of the d -cube on π .

Let also p_i and p_j be the i^{th} and j^{th} points with respect to the axis 0 in a d -permutation σ . The points p_i and p_j are k -adjacents if one has $\sigma_k(p_i) = \sigma_k(p_j) \pm 1$. Additionally, p_i and p_j are 0-adjacent if $i = j \pm 1$.

Definition 3. A *generalised vincular pattern* $\pi|_{X_0, \dots, X_{d-1}}$ is a d -permutation π of size k along with a list of *adjacencies* given by subsets of $[k-1]$. Given a d -permutation σ , the set of points $p_1 \dots p_k$ is an occurrence of $\pi|_{X_0, \dots, X_{d-1}}$ if it is an occurrence of π and

⁵This presentation is based on [1].

if it satisfies that for any j in any X_k the j^{th} and the $j + 1^{th}$ point (of the occurrence) with respect to the axis k are k -adjacent.

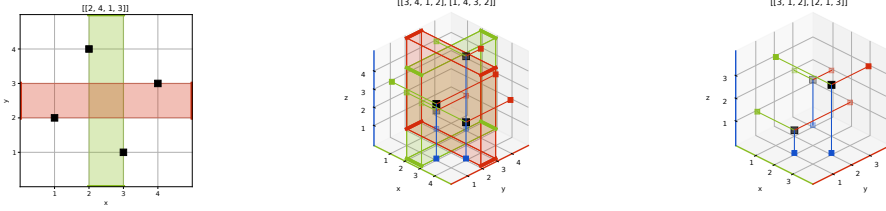


Figure 15: The patterns $2413|_{2,2}$, $(3412, 1432)|_{2,2}$, and $(312, 213)$.

In [3] and [4], a definition of *separable d -permutations* and of *Baxter d -permutations* was proposed. It was shown that these generalizations were characterized by the forbidden patterns $\text{Sym}(2413), \text{Sym}((312, 213))$ for separable d -permutations and by the forbidden patterns $\mathfrak{S}(2413|_{2,2})$, $\mathfrak{S}((312, 213)|_{1,2})$, $\mathfrak{S}((3412, 1432)|_{2,2})$, $\mathfrak{S}((2143, 1423)|_{2,2})$ for Baxter d -permutations.

Higher dimensional floorplans

A *floorplan* of size n is a partition of a rectangle using n interior-disjoint rectangles. These combinatorial objects have been studied in various fields of computer science, architecture or discrete geometry.

A *d -dimensional floorplan* (or a *box partition*) is a partition of a d -dimensional hyperrectangle with n interior-disjoint d -dimensional hyperrectangles (called *blocks*). A $(d - 1)$ -hyperrectangles with fixed coordinate x_i is called *facet* of axis i . Given an axis i , the boundary of a d -dimensional hyperrectangle defines two facets of such axis. A *d -dimensional floorplan* is *generic* if the set of facets that share the same i -th coordinate is a single facet. A *border* is a maximal facet of the interior of the bounding d -hyperrectangle.

Definition 4. A *d -floorplan* is a generic d -dimensional floorplan that has no border crossing.

Two d -floorplans are equivalent if the relative positions (up, down, left, right etc...) of their boxes are the same. Figure 16 (on the left) shows an example of 3-floorplan. Mosaic floorplans (or 2-floorplans) are known to be in bijection with Baxter permutations [2]. This bijection also defines a bijection between separable permutations and a subset of mosaic floorplans, called *guillotine partitions*.

In [3], a generalization to arbitrary dimensions of guillotine partitions, was considered. The authors find a bijection between 2^{d-1} -dimensional guillotine partitions and separable d -permutations.

In the work presented here, we introduce a generalization of the bijection [2] between mosaic floorplans and Baxter permutations which also generalizes the one in [3].

Theorem 5. *There exists a bijection between d -permutations with n points avoiding $Sym((\textcolor{blue}{312}, \textcolor{red}{213}))$, $\mathfrak{S}(2413|_{2,2})$ and 2^{d-1} -floorplans with n blocks.*

The multipermutations involved in the bijection are defined by the avoidance of the vincular patterns of Baxter permutations and of the dimension 3 patterns of the separable d -permutations. This set is strictly included in the set of Baxter d -permutations defined in [4]. Table 1 summarizes the different objects and related permutations classes.

Objects	Permutations	Pattern avoidance
Slicing floorplans	Separable	$Sym(2413)$
Mosaic floorplans	Baxter	$\mathfrak{S}(2413 _{2,2})$
2^{d-1} -guillotine floorplans	d -Separable	$Sym(2413), Sym((\textcolor{blue}{312}, \textcolor{red}{213}))$
2^{d-1} -Floorplans	sub d -Baxter	$\mathfrak{S}(2413 _{2,2}), Sym((\textcolor{blue}{312}, \textcolor{red}{213}))$
	d -Baxter	$\mathfrak{S}(2413 _{2,2}), \mathfrak{S}((\textcolor{blue}{312}, \textcolor{red}{213}) _{1,2,.}),$ $\mathfrak{S}((\textcolor{blue}{3412}, \textcolor{red}{1432}) _{2,2,.}),$ $\mathfrak{S}((\textcolor{blue}{2143}, \textcolor{red}{1423}) _{2,2,.})$

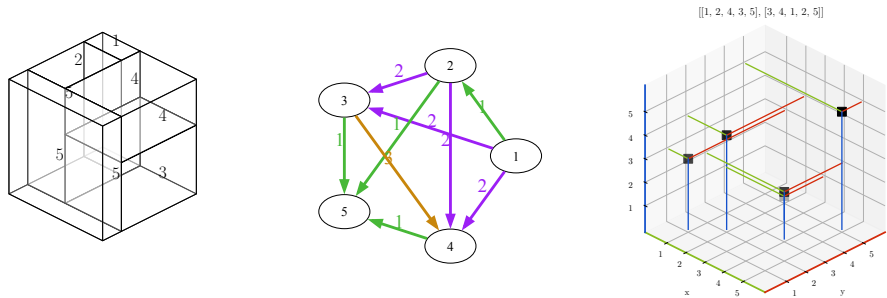


Figure 16: On the left an example of a 3–floorplan. In the middle the relative order of each blocks with respect to each direction (x, y, z) . On the right the corresponding 3-permutation (considering the 3-floorplan as a 4-floorplan).

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This talk is based on joint work with Christian Bean and Paul C. Bell

A *Cayley permutation* is a word $\pi \in \mathbb{N}^*$ such that every number between 1 and the maximum value of π appears at least once. Cayley permutations can be seen as a generalisation of permutations where repeated values are allowed. Definitions of pattern containment and Cayley permutation classes follow the same ideas as defined for permutations where the patterns contained are also Cayley permutations, so the Cayley permutation class $\text{Av}(11)$ describes all permutations. Recently, Cayley permutation classes avoiding Cayley permutations of length 3 have been enumerated using species by Claesson, Cerbai, Ernst and Golab [1] as well as in Golab's thesis [2].

The set of Cayley permutations is in bijection with the set of ordered set partitions; the value i in the j^{th} block of the ordered set partition implies the i^{th} index of the Cayley permutation has value j . For example, the Cayley permutation 21321 corresponds to the ordered set partition $\{2, 5\}, \{1, 4\}, \{3\}$. Therefore, the set of Cayley permutations are counted by the Fubini numbers.

The goal of this work is to utilise some of the methods that have been developed for permutation classes to study Cayley permutation classes. In particular, we extend the insertion encoding of permutations introduced by Albert, Linton and Ruskuc [3] to Cayley permutations in two different ways. First, by inserting new maxima, which we will call the *vertical insertion encoding* and then by inserting new rightmost values, which we will call the *horizontal insertion encoding*. As permutations are closed with respect to the inverse symmetry, inserting a new maximum is equivalent to inserting a new rightmost value so these two methods are equivalent. For Cayley permutations this is no longer the case.

Our main result is to fully classify the Cayley permutation classes that have a regular insertion encoding, which follows a similar theory to the theory for permutations. A size $n + k$ Cayley permutation π is a *vertical juxtaposition* of the size n Cayley permutation σ and size k Cayley permutation τ if the smallest n values of π standardise to σ and the largest k values standardise to τ . For example, a vertical juxtaposition of 12432 and 221 is 16624352.

Similarly, a size $n + k$ permutation π is a *horizontal juxtaposition* of the size n permutation σ and the size k permutation τ if the size n prefix of π standardises to σ and the size k suffix of π standardises to τ . For example, 13245 is a juxtaposition of the increasing sequence 12 and the increasing sequence 123.

There are nine classes of vertical juxtapositions which are central to our theory. These arise from vertically juxtaposing strictly increasing, strictly decreasing or constant sequences. We will use $\mathcal{V}_{a,b}$ with $a, b \in \{D, I, C\}$ to denote the Cayley permutation class which is the vertical juxtaposition of a strictly increasing sequence if $a = I$, strictly decreasing sequence if $a = D$ and constant sequence if $a = C$ with a strictly

increasing sequence if $b = I$, strictly decreasing sequence if $b = D$ and constant sequence if $b = C$. For example, the Cayley permutation 23141 belongs to the class $\mathcal{V}_{C,I}$. We also define the four classes of horizontal juxtapositions that come from horizontally juxtaposing strictly increasing and strictly decreasing sequences as $\mathcal{H}_{I,I}$, $\mathcal{H}_{I,D}$, $\mathcal{H}_{D,I}$, and $\mathcal{H}_{D,D}$. With these definitions we can state our main results.

Theorem 1. *A Cayley permutation class $\text{Av}(B)$ has a regular vertical insertion encoding if and only if B contains a Cayley permutation from each of the nine classes $\mathcal{V}_{I,I}$, $\mathcal{V}_{I,D}$, $\mathcal{V}_{I,C}$, $\mathcal{V}_{D,I}$, $\mathcal{V}_{D,D}$, $\mathcal{V}_{D,C}$, $\mathcal{V}_{C,I}$, $\mathcal{V}_{C,D}$, and $\mathcal{V}_{C,C}$ of vertical juxtapositions.*

Theorem 2. *A Cayley permutation class $\text{Av}(B)$ has a regular horizontal insertion encoding if and only if it B contains a Cayley permutation from each of the four classes $\mathcal{H}_{I,I}$, $\mathcal{H}_{I,D}$, $\mathcal{H}_{D,I}$ and $\mathcal{H}_{D,D}$ of horizontal juxtapositions.*

Checking if a Cayley permutation class has either type of regular insertion encoding can, therefore, be done using a linear time algorithm. Table 7 shows the number of Cayley permutation classes defined by avoiding size 3 patterns with vertical or horizontal insertion encoding that is regular. Every class defined by avoiding nine or more size three patterns has a regular insertion encoding of either type. Of the 8191 bases containing Cayley permutations of length 3 only, 7914 of them have a regular insertion encoding.

Size of basis	Number of classes	Regular vertical insertion encoding	Regular horizontal insertion encoding	Either
1	13	0	0	0
2	78	0	0	0
3	286	67	45	85
4	715	377	294	441
5	1287	960	846	1050
6	1716	1508	1433	1582
7	1716	1631	1607	1673
8	1287	1267	1264	1281
9	715	713	713	715

Table 7: The number of Cayley permutation classes defined by avoiding size 3 patterns that have a regular insertion encoding.

We also have a method for computing the regular vertical insertion encodings, which can be adapted to the horizontal case. Vatter [4] gave a method for constructing a DFA for the insertion encoding of slot-bounded permutation classes. The method from that paper can be extended to slot-bounded Cayley permutation classes. However, we present an alternative approach that is more efficient for permutations and can be applied to Cayley permutations. Our algorithm follows methods similar to those introduced by Albert, Bean, Claesson, Nadeau, Pantone, and Ulfarsson [5] for permutations, but adapted to Cayley permutations.

Cerbai [6] found the basis for hare pop-stack sortable Cayley permutations to be $\text{Av}(231, 312, 2121)$ and left their enumeration as an open problem. Using our algo-

rithm, we present the generating function $S(x)$ for $\text{Av}(231, 312, 2121)$ to be

$$S(x) = \frac{2x^2 - 2x^3 - x}{4x^3 - 6x^2 + 5x - 1},$$

for the sequence beginning 1, 3, 11, 41, 151, 553, 2023 which is sequence A335793 in the Online Encyclopedia of Integer Sequences (OEIS) [8].

Cayley permutations are in bijection with ordered set partitions. The unordered set partitions are in bijection with a subset of Cayley permutation, referred in the literature as *restricted growth functions* [7]. Restricted growth functions are the Cayley permutations with the additional condition that for all values k and ℓ in the Cayley permutation, if $k < \ell$ then the first occurrence of k appears before the first occurrence of ℓ . Our algorithm can be adapted to compute the rational generating functions for pattern-avoiding restricted growth functions with either a horizontal or vertical regular insertion encoding. It should be possible to classify these regular cases in a similar way as we have done for Cayley permutations, but this is left for future work.

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AN OPTIMAL ALGORITHM FOR SORTING PATTERN-AVOIDING SEQUENCES

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A standard argument shows that any comparison-based algorithm for sorting a set of permutations Γ , each of length n , must make at least $\Omega(\log |\Gamma| + n)$ comparisons in the worst case. In order to see how this lower bound applies to pattern-avoiding sequences, it invites the natural question of how many pattern-avoiding permutations of given length are there. In fact, it has been a long-standing open problem whether the number of π -avoiding permutations of length n grows at most exponentially in n . This question, known as the *Stanley-Wilf conjecture*, was eventually answered positively by Marcus and Tardos [8] building on the works of Klazar [7] and Füredi and Hajnal [6]. In fact, the limit $s_\pi = \lim_{n \rightarrow \infty} \sqrt[n]{|\text{Av}_n(\pi)|}$ exists [1], where $\text{Av}_n(\pi)$ denotes the set of all π -avoiding permutations of length n ; s_π is known as the *Stanley-Wilf limit* of π . It follows that any comparison-based algorithm for sorting π -avoiding inputs must make at least $\Omega((\log s_\pi + 1) \cdot n)$ comparisons in the worst case.

Our main result is an asymptotically optimal sorting algorithm for pattern-avoiding sequences that matches this information-theoretic lower bound for each π up to a global constant. Furthermore, the algorithm requires prior knowledge of neither the forbidden pattern π nor its Stanley-Wilf limit s_π .

Theorem 1. *There is an algorithm that sorts a π -avoiding sequence of length n in $O((\log s_\pi + 1) \cdot n)$ time even if π is a priori unknown.*

The algorithm is built from two components. The first component is an efficient multi-way merge procedure capable of merging up to $\frac{n}{\log n}$ sequences with n elements in total in linear time, as long as the sequences originate from a π -avoiding input. Its analysis is heavily based on the Marcus-Tardos theorem [8].

The second component is an efficient algorithm to sort a large set of short sequences. Naturally, many sequences in such a set must be pairwise order-isomorphic, which suggests that we could try grouping them into equivalence classes and then actually sorting only one sequence from each class. Unfortunately, computing the equivalence classes is in some sense as hard as sorting itself. Instead, we take a different approach of guessing a shallow decision tree for sorting π -avoiding permutations guaranteed by the result of Fredman [5].

Theorem 1 then follows by first cutting up the input into sequences of length roughly $\log \log \log n$ which we can efficiently sort using the optimal decision tree. The sorted sequences then serve as a basis for a bottom-up merge sort using the efficient multi-way merge.

Furthermore, let us point out that Theorem 1 achieves the information-theoretic lower bound for arbitrary π -avoiding sequences, not only permutations. This is in sharp contrast with previous works [3, 2] where the generalization beyond permutations comes with a price of worsening the dependence on π . In order to achieve this,

we strengthen a combinatorial result of Cibulka [4] and show that the number of π -avoiding $n \times n$ binary matrices with n 1-entries is at most $c_\pi^{2n+O(1)}$.

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Let us take a permutation property that can be defined by predicate logic. For instance, any pattern avoidance is such a property. We are interested in the density of the permutations in a given class that satisfy that property as the size of them goes to infinity. If there exists a limit for the density for any property, then we say that the class of permutations is first-order stable. We show that $\text{Av}(\mathbf{321})$ is first-order stable.

Given $\sigma \in S_n$, we define two binary relations on $\{1, 2, \dots, n\}$, (1) position ($<_P$) and (2) value ($<_V$) as follows:

$$\begin{aligned} i <_P j & \text{ if and only if } i < j, \\ i <_V j & \text{ if and only if } \sigma(i) < \sigma(j). \end{aligned}$$

So, any permutation in S_n can be viewed as a structure with domain $A = \{1, 2, \dots, n\}$ and a set of binary relations $R = \{<_P, <_V\}$. Then, for example, we can express the **321**-pattern avoidance as

$$\varphi_1 = \neg [\exists x \exists y \exists z [(x <_P y) \wedge (y <_P z) \wedge (y <_V x) \wedge (z <_V y)]],$$

and we say $\sigma \models \varphi_1$ if and only if $\sigma \in \text{Av}(\mathbf{321})$. Since we are interested in the limiting density, defining $P(A)$ as the probability of an event A , we have

$$P(\sigma_n \models \varphi_1) = \frac{C_n}{n!}$$

where σ_n is a randomly chosen permutation in S_n and C_n is the n th Catalan number. A second example is the sentence that "there exists an inversion", which can be symbolized as

$$\varphi_2 = \exists x \exists y [(x <_P y) \wedge (y <_V x)].$$

Our main result reads

Theorem 1. [4] *Let $\sigma_n \in \text{Av}_n(\mathbf{321})$ and φ be a first-order sentence on permutations. Then*

$$\lim_{n \rightarrow \infty} P(\sigma_n \models \varphi) \text{ exists.}$$

The limit law for $\text{Av}(\mathbf{231})$ is proven in [1] using the recursive structure of **231**-avoiding permutations. The structural differences between $\text{Av}(\mathbf{231})$ and $\text{Av}(\mathbf{321})$ do not allow their methods to be extended to the latter case. We, on the other hand, use random processes to exploit the representability of permutations in $\text{Av}(\mathbf{321})$ by two increasing subsequences. See the figure below.

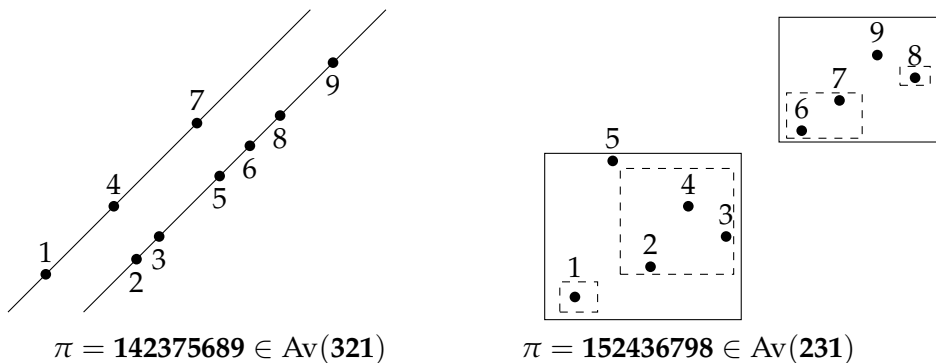


Figure 17: A pair of increasing subsequences vs. a recursive pattern

Random processes started to be used in this context very recently only with a few exceptions. In addition to the permutation classes, the logic of random graphs and random groups can be studied with analogous methods. Out of necessity, our method differs from the earlier ones, two of which are mentioned in the table below. It relies on an operator view which can be viewed as an abstraction of the processes governed by Markov chains.

Random process	Finite Markov chains	Countable Markov chains	Symbolic chains	Operator viewpoint
Example	Layered permutations [2]	Mallows random permutations [3]	No example studied	$\text{Av}(321)$ [4]
Conditions on the matrix (operator)	stochastic irreducible aperiodic	stochastic irreducible aperiodic pos. recurrent	non-negative irreducible aperiodic pos. recurrent	(connected) (non-partite) (compact)

Table 8: Applications of random processes to show first-order stability of some permutation classes

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This talk is based on joint work with Christian Bean

A *tiling* is a way of representing a set of gridded permutations that is defined by the avoidance of some gridded patterns (obstructions) and the containment of others (requirements). Tilings were introduced by Albert, Bean, Claesson, Nadeau, Pantone, and Ulfarsson [ABC⁺25] in order to algorithmically discover the structure of permutation classes using a method called Combinatorial Exploration.

The figure below shows, on the left, a tiling with obstruction patterns shown in red with circle points and solid edges and requirement patterns shown in blue with square points and dashed edges. This tiling represents all 2×3 gridded permutations that avoid 132 in cell $(0,0)$ and in cell $(2,0)$, avoid the pattern 123 with the 1 in cell $(0,0)$ and the 2 and the 3 in cell $(2,0)$, and contain exactly one point in cell $(1,1)$. The tiling on the right is just visually simplified, representing the same set.



Figure 18: ([ABC⁺25]) On the left, a tiling with all obstructions and requirements depicted. On the right, the same tiling shown using visual shortcuts.

We show in this talk that tilings have utility beyond their role in Combinatorial Exploration. We will present an algorithm that uses tilings to describe how permutations in any given class can be built by inserting new maximum entries into existing permutations in the class. This is essentially the *insertion encoding* of Albert, Linton, and Ruškuc [ALR05] and of Vatter [Vat12], using tilings to keep track of where new entries can be inserted. This is also similar to the method used by Conway, Guttmann, and Zinn-Justin [CGZJ18] to compute the number of 4231-avoiding permutations up length 50, except that it can be used on any permutation class, and does not require any up-front work of understanding the structure of the class.

For some permutation classes, our method automatically discovers a polynomial-time counting algorithm. For others it is exponential, but with a lower base than the growth rate of the class. The method is also easily parallelizable.

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CHARACTERIZATION OF AVOIDANCE OF ONE-SIDED 2- AND 3- SEGMENT PATTERNS IN RECTANGULATIONS BY MESH PATTERNS

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This talk is based on joint work with Andrei Asinowski, Torsten Mütze, and Namrata

Pattern avoidance is a fundamental concept in combinatorics, and in this work, we consider pattern avoidance in rectangulations. A *rectangulation* $\mathcal{R}$ is a decomposition of a rectangle R into a finite number of rectangles such that there are no $+$ joints. The *size* of a rectangulation is the number of rectangles in the decomposition; it is denoted by n . A *segment* of a rectangulation is a maximal straight line that consists of one or several sides of rectangles of $\mathcal{R}$, and is not included in one of the sides of R .

In order to consider equivalence classes of rectangulations, we introduce *left-right* and *above-below* relations:

- Rectangle r is *left of* rectangle r' (equivalently, r' is *right of* r) if there is a sequence of rectangles, $r = r_1, r_2, \dots, r_k = r'$ such that the right side of r_i and the left side of r_{i+1} lie in the same segment for $i = 1, 2, \dots, k-1$.
- Rectangle r is *below* rectangle r' (equivalently, r' is *above* r) if there is a sequence of rectangles, $r = r_1, r_2, \dots, r_k = r'$ such that the upper side of r_i and the bottom side of r_{i+1} lie in the same segment for $i = 1, 2, \dots, k-1$.

NW-SE labeling We label the rectangles from 1 to n according to the NW-SE order described in [3]. The rectangle with label j ($1 \leq j \leq n$) is denoted by r_j . The rectangle r_1 contains the NW (top-left) corner of R and r_n contains the SE (bottom-right) corner of R . If j is fixed, then r_i with $i < j$ are precisely the rectangles above or to the left of r_j , and r_i with $i > j$ are precisely the rectangles below or to the right of r_j . For the detailed algorithm of NW-SE labeling of rectangles in a rectangulation, see [3].

Weak and strong equivalences There are two natural ways to define combinatorial equivalence for rectangulations: *weak equivalence* which preserves left-right and above-below relations between rectangles, and *strong equivalence* which also preserves contact between rectangles. Equivalence classes are called *weak rectangulations* and *strong rectangulations* and the set of all rectangulations of size n is denoted by R_n^w or R_n^s , respectively. Each weak rectangulation has a representative in which each rectangle meets the NW-SE diagonal, this representative is called a *diagonal rectangulation*. See Figure 19 for an example.

Representing rectangulations as posets and permutations Given any rectangulation $\mathcal{R}$, we can encode the contact between rectangles by a poset, called the adjacency poset and denoted $P_a(\mathcal{R})$. When the rectangulation is diagonal, we call it the weak poset and denote it $P_w(\mathcal{R})$. This is the weak poset for the entire weak rectangulation equivalence class. This poset was introduced by Law and Reading in [4], where they also described a mapping γ_w from the set of permutations S_n to the set of weak rectangulations R_n^w . In [3], Asinowski et al. generalized the adjacency poset to strong

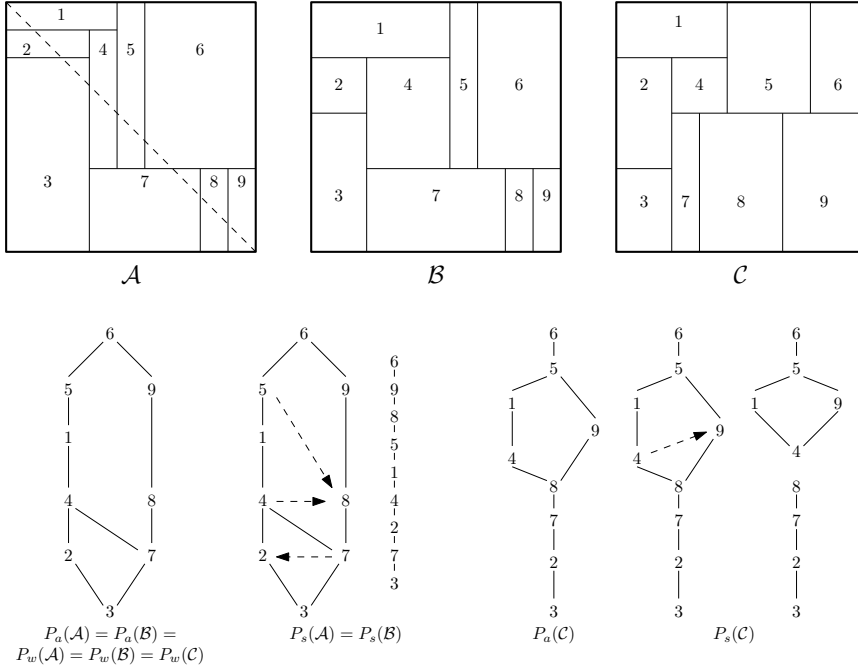


Figure 19: $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are all weakly equivalent, only $\mathcal{A}$ and $\mathcal{B}$ are strongly equivalent. $\mathcal{A}$ is a diagonal rectangulation. The adjacency poset for $\mathcal{A}$ and $\mathcal{B}$, and the weak poset for all of three is the same. The special relations added to form the strong poset from the adjacency poset are shown with dashed arrows.

rectangulations by also considering certain “special relations” when constructing the poset. That is, they begin with the adjacency poset for a strong rectangulation $\mathcal{R}$, and then add in additional relations based on certain rules. They also described a map γ_s from S_n to R_n^s . Finally, they proved that $\gamma_w^{-1}(\mathcal{R}) = \mathcal{L}(P_w(\mathcal{R}))$ and $\gamma_s^{-1}(\mathcal{R}) = \mathcal{L}(P_s(\mathcal{R}))$ where $\mathcal{L}(P)$ denotes the set of linear extensions of a poset P .

Pattern avoidance in rectangulations A rectangulation $\mathcal{R}$ contains a pattern p if there is an injection from the segments of (any representative of) p into the segments of (any representative of) $\mathcal{R}$ which preserves incidences, orientations, and order of incidence of neighbors of the segments of p . Otherwise, $\mathcal{R}$ avoids p . Following recent work, there has been growing interest in studying pattern avoidance in rectangulations, as they are highly versatile objects that can have connections to many other combinatorial structures [1, 2, 3]. In our work, we restrict to rectangulation patterns with two or three segments in which no segment has neighbors on both sides (“one-sided”). We give one-to-one mappings between rectangulation patterns and mesh permutation patterns such that a rectangulation $\mathcal{R}$ avoids the given rectangulation pattern if and only if the linear extensions of its poset $\mathcal{L}(P(\mathcal{R}))$ avoid the corresponding permutation patterns. The question of representing rectangulation patterns by permutation patterns was given as an interesting direction for future research by Merino and Mütze [5]. Our results are summarized for all one-sided 2- and 3-segment patterns in the table.

Rectangulation Pattern (p)	Permutation Mesh Pattern (τ_p)	weak/ strong	Enumeration for $n = 1, \dots, 10$	OEIS
		weak	1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796	A000108
		strong	1, 2, 5, 15, 51, 189, 746, 3091, 13311, 59146	A279555
		weak	1, 2, 6, 21, 81, 334, 1445, 6485, 29954, 141609	
		strong	1, 2, 6, 23, 103, 514, 2785, 16097, 98030, 623323	
		weak	1, 2, 6, 21, 79, 309, 1237, 5026, 20626, 85242	A026737 & A111279
		strong	1, 2, 6, 23, 101, 482, 2433, 12787, 69270, 384134	
		weak	1, 2, 6, 21, 80, 322, 1347, 5798, 25512, 114236	A106228
		strong	1, 2, 6, 23, 102, 499, 2622, 14547, 84229, 504775	
		weak	1, 2, 6, 21, 80, 322, 1347, 5798, 25512, 114236	A106228
		strong	1, 2, 6, 23, 102, 499, 2622, 14547, 84229, 504775	

Our results For each rectangulation pattern p and associated permutation mesh pattern τ_p given in the table, we prove the following statement:

A permutation $\pi \in S_n$ avoids τ_p if and only if $\gamma_s(\pi)$ avoids p .

For each result we follow the same proof outline: we begin by proving that $\pi \in S_n$ avoids τ_p if and only if $\gamma_w(\pi)$ avoids p . Hence, given a weak rectangulation $\mathcal{R}$ which avoids p , every linear extension of $P_w(\mathcal{R})$ avoids τ_p . As $P_s(\mathcal{R})$ contains all of the relations included in $P_w(\mathcal{R})$, $\mathcal{L}(P_s(\mathcal{R})) \subseteq \mathcal{L}(P_w(\mathcal{R}))$, so we conclude the desired result.

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ROBINSON–SCHENSTED SHAPES ARISING FROM CYCLE DECOMPOSITIONS

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This talk is based on joint work with Martha Du Preez, William Erickson, Jonathan Feigert, Markus Hunziker, Jonathan Meddaugh, Mitchell Minyard, and Kyle Rosengartner

Statement of the Problem. At the heart of classical algebraic combinatorics is the representation theory of the symmetric group S_n . In turn, much of this theory can be expressed in terms of integer partitions. In this work, we describe the subtle relationship between two partitions closely associated with each element $\sigma \in S_n$: the *cycle type* of σ , on one hand, and the *shape* of σ via the Robinson–Schensted correspondence on the other hand. Although separately each of these partitions is fundamental to the general theory, the two had not yet been studied *together* until a very recent paper [1] treating the special case where σ is a cyclic (or almost cyclic) permutation. (The most closely related works study cycle types and descents [2], or shapes and inversions [3].) A natural question is the following: which shapes arise from the elements of a given cycle type?

It is well known that the conjugacy classes of S_n (and also its irreducible complex representations) can be naturally labeled by the integer partitions α of n , written as $\alpha \vdash n$. In particular, the conjugacy class of $\sigma \in S_n$ is labeled by the partition $\alpha = (\alpha_1, \dots, \alpha_r)$ giving the *cycle type* of σ , which is easily read off from the expression of σ in disjoint cycle notation: $\sigma = (\alpha_1\text{-cycle})(\alpha_2\text{-cycle}) \cdots (\alpha_r\text{-cycle})$. Throughout the paper, we write $\mathcal{C}_\alpha$ to denote the conjugacy class of S_n consisting of elements with cycle type α .

Another key concept in the representation theory of S_n and in algebraic combinatorics in general is the Robinson–Schensted (RS) correspondence. The RS correspondence is a bijection

$$S_n \xrightarrow{\text{RS}} \coprod_{\lambda \vdash n} \text{SYT}(\lambda) \times \text{SYT}(\lambda),$$

where $\text{SYT}(\lambda)$ denotes the set of standard Young tableaux with shape λ , meaning that the partition λ gives the row lengths of the tableaux. If the RS correspondence takes σ to a pair $(P, Q) \in \text{SYT}(\lambda) \times \text{SYT}(\lambda)$, then we say that λ is the *RS shape* of σ , which we denote by writing $\text{sh}(\sigma) = \lambda$. Thus in the example

$$\sigma = (3, 5, 4, 7)(1, 2, 6) \xrightarrow{\text{RS}} \left(\begin{array}{|c|c|c|} \hline 1 & 3 & 7 \\ \hline 2 & 4 & \\ \hline 5 & & \\ \hline 6 & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 7 & \\ \hline 5 & & \\ \hline 6 & & \\ \hline \end{array} \right), \quad (12)$$

we have $\sigma \in \mathcal{C}_{(4,3)}$, and $\text{sh}(\sigma) = (3, 2, 1, 1)$. The main problem of our work is to describe the elements of

$$\mathcal{S}_\alpha := \{\text{sh}(\sigma) : \sigma \in \mathcal{C}_\alpha\}.$$

Main Result. As a preliminary result, for all $\alpha = (\alpha_1, \dots, \alpha_r) \vdash n$, we prove that the

partitions in $\mathcal{S}_\alpha$ have Young diagrams fitting inside a certain bounding box:

$$\mathcal{S}_\alpha \subseteq \mathcal{B}_\alpha, \quad (13)$$

where

$$\mathcal{B}_\alpha := \left\{ \lambda \vdash n : \begin{array}{l} \lambda \text{ has at most} \\ (n - r + \#\{i : \alpha_i = 2\} + \delta_{1,\alpha_r}) \text{ many rows and} \\ (n - r + \#\{i : \alpha_i = 1\}) \text{ many columns} \end{array} \right\}. \quad (14)$$

For example, if $\alpha = (4, 2)$, then $\mathcal{B}_\alpha$ consists of all partitions of $n = 6$ whose Young diagram fits inside the box of dimensions $(6 - 2 + 1 + 0) \times (6 - 2 + 0) = 5 \times 4$. Concretely, we have

$$\mathcal{B}_{(4,2)} = \left\{ \begin{array}{c} \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \square & & & \\ \hline \end{array} \end{array} \right\}.$$

For certain cycle types α , the containment in (13) is, in fact, an equality. We can thus reframe our main problem as follows: classify the cycle types α such that $\mathcal{S}_\alpha = \mathcal{B}_\alpha$, and for the remaining cycle types α , determine the complement $\mathcal{B}_\alpha \setminus \mathcal{S}_\alpha$. The following theorem is the main result of our paper, where we solve the problem in the case $r = 2$, that is, where $\alpha = (\alpha_1, \alpha_2)$.

Theorem 1. *Let n be a positive integer, and let $\alpha = (\alpha_1, \alpha_2) \vdash n$.*

1. *If n is odd, then $\mathcal{S}_\alpha = \mathcal{B}_\alpha$.*
2. *If n is even, then $\mathcal{S}_\alpha = \mathcal{B}_\alpha$ unless α occurs in the following table:*

α	$\mathcal{B}_\alpha \setminus \mathcal{S}_\alpha$
$(n - 1, 1)$	$\{(\frac{n}{2}, \frac{n}{2})\}$
$(\frac{n}{2}, \frac{n}{2})$, where $4 \mid n$	$\{(n - 2, 1, 1), (3, 1, \dots, 1)\}$
$(\frac{n}{2}, \frac{n}{2})$, where $4 \nmid n$	$\{(n - 2, 1, 1)\}$
$(4, 2)$	$\{(2, 2, 2)\}$
$(5, 3)$	$\{(2, 2, 2, 2)\}$

Our Theorem 1 generalizes the main result of [1], which can be restated as follows, using the language of the present paper:

1. If n is odd, then $\mathcal{S}_{(n)} = \mathcal{B}_{(n)}$ and $\mathcal{S}_{(n-1,1)} = \mathcal{B}_{(n-1,1)}$.
2. If n is even, then $\mathcal{S}_{(n)} = \mathcal{B}_{(n)}$ and $\mathcal{B}_{(n-1,1)} \setminus \mathcal{S}_{(n-1,1)} = \{(\frac{n}{2}, \frac{n}{2})\}$.

We emphasize that by introducing the bounding box $\mathcal{B}_\alpha$, we are able to state these results in a uniform manner, thus avoiding the need to designate certain “trivial shapes” and to make exceptions for small values of n .

Admissible Tableaux and α -colorings. In order to prove Theorem 1, we introduce two key combinatorial objects, which we call *admissible tableaux* and α -colorings. We say a standard tableau is *admissible* if it remains standard when justified along the bottom (i.e., when acted on by “gravity”). It turns out that the admissibility of a tableau Q is equivalent to the following: for any tableau P of the same shape, the permutation $RS^{-1}(P, Q)$ can be read off from P by following the order of the entries in the vertical reflection of Q , which we denote by $Q^\uparrow$. In turn, for an admissible $Q \in \text{SYT}(\lambda)$, an α -coloring of $Q^\uparrow$ is a coloring of its entries which, via a certain canonical spiral construction, produces a permutation $\sigma \in \mathcal{C}_\alpha$ with $\text{sh}(\sigma) = \lambda$. This σ cyclically permutes the entries of each color in $Q^\uparrow$ to obtain a standard P such that $\sigma = RS^{-1}(P, Q)$. The crux of the paper is therefore where we construct α -colorings for all shapes λ , with the exception of the pairs (α, λ) given above in the table in Theorem 1. (There is also one family (α, λ) such that $\lambda \in \mathcal{S}_\alpha$ but no α -coloring exists; in this case we exhibit the requisite σ directly.)

Open Problems and Conjectures. As a first step toward extending the results in this paper to generic cycle types $(\alpha_1, \dots, \alpha_r)$ where $r > 2$, we point out a special case, following from a result of Schützenberger relating to involutions, in which we can describe $\mathcal{S}_\alpha$ explicitly, namely when $\alpha_1 \leq 2$:

$$\mathcal{S}_{(2^{r-k}, 1^k)} = \{\lambda \vdash n : \lambda \text{ has exactly } k \text{ many columns of odd length}\}.$$

Outside of this special case described above, however, for $r > 2$, an explicit and comprehensive description of $\mathcal{S}_\alpha$ quickly becomes quite complicated. Somewhat surprisingly, the source of these complications lies entirely in the presence of repeated values among the α_i ’s. In fact, we conjecture that for $r > 2$, we have $\mathcal{S}_\alpha = \mathcal{B}_\alpha$ whenever $\alpha_1 > \dots > \alpha_r > 1$. (It would then follow that for a fixed r , as $n \rightarrow \infty$, the proportion of cycle types satisfying $\mathcal{S}_\alpha = \mathcal{B}_\alpha$ approaches 100%.) See the last section of our paper for additional details, along with further conjectures and open problems concerning cycle types and α -colorings.

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This talk is based on joint work with Michael Albert and Dominic Searles

Introduction

In [1], Bloom and Sagan introduce a *partial shuffle* as a tool for proving pattern avoidance results in the context of quasisymmetric functions. They show that taking this partial shuffle as a basis generates an infinite family of Wilf-equivalent permutation classes.

Our work expands on this result, first with a reinterpretation of the proof of Wilf-equivalence for these partial shuffle classes. We then show that Wilf-equivalence is preserved upon including a descending permutation of arbitrary length in the partial shuffle basis, giving a wider family of Wilf-equivalent bases. We then provide some enumerative results, including demonstrating the appearance of the Catalan numbers in the leading coefficient of the enumerating polynomial for these classes.

Results

Let $[\ell] = \{1, 2, \dots, \ell\}$, and for fixed ℓ , choose $a \in [\ell]$ and b such that $a + b = \ell$. Then, the partial shuffle $\Pi(a, b)$ consists of all permutations of $[\ell]$ where every element from $[\ell] - \{a\}$ is in increasing order, apart from the strictly increasing permutation $12 \dots \ell$.

Example 1.

$$\Pi(2, 3) = \{13452, 13425, 13245, 21345\}.$$

The following is Lemma 4.4 in [1]:

Lemma 2. *For pairs of natural numbers (a, b) and (c, d) such that $a + b = c + d$, the pattern avoidance classes $Av(\Pi(a, b))$ and $Av(\Pi(c, d))$ are Wilf-equivalent.*

Our proof is visual and involves defining a function S which acts on $\pi \in Av(\Pi(a, b))$, rotating certain subsets of elements to produce, after finitely many iterations, a permutation $S(\pi) \in Av(\Pi(a - 1, b + 1))$. We show that this function is injective, and that it can be reversed.

Letting δ_k denote the descending permutation of length k , we prove the following:

Theorem 3. *If δ_k is the longest descending subpermutation of $\pi \in Av(\Pi(a, b))$, then δ_k is the longest descending subpermutation in $S(\pi)$.*

An immediate corollary is that if $a + b = c + d$, then for any $k \geq 1$, $\Pi(a, b) \cup \delta_k$ is Wilf-equivalent to $\Pi(c, d) \cup \delta_k$. We determine the approximate growth of these classes:

Theorem 4. For a, b such that $a + b = \ell$, the permutations of length n in the pattern avoidance class $Av(\Pi(a, b) \cup \delta_k)$ are counted by a polynomial in n , with degree $(\ell - 2)(k - 2)$.

Proposition 5. For a, b such that $a + b = \ell$, and $\ell \geq 3$, the polynomial $p(n)$ which counts $\#Av_n(\Pi(a, b) \cup \delta_3)$ has leading term $C_{\ell-2} \binom{n}{\ell-2}$, where $C_{\ell-2}$ is the $(\ell - 2)^{th}$ Catalan number.

There does not appear to be a nice closed form for the other terms in the counting polynomials, in general, although automatic enumeration schemes [2] do exist for polynomial classes such as these. With the basis $\Pi(a, b) \cup \delta_3$, we conjecture that:

Conjecture 6. For fixed a, b , where $\ell = a + b \geq 3$, and $n \geq 2(\ell - 2) + 1$,

$$\#Av_n(\Pi(a, b) \cup \delta_3) = C_{\ell-2} \binom{n}{\ell-2} - \sum_{1 \leq h < n-2} T_{\ell-2, h} \binom{n}{\ell-3-h},$$

where the coefficients $T_{\ell-2, h}$ correspond to rows of the transposed Catalan triangle;

$$T_{p, q} = \frac{q \binom{2p-q}{p}}{2p - q}$$

(OEIS sequence A033184).

Example 7. For $n \geq 13$, and $a + b = 9$, $\#Av_n(\Pi(a, b) \cup \delta_3)$ is given by

$$429 \binom{n}{7} - 132 \binom{n}{5} - 132 \binom{n}{4} - 90 \binom{n}{3} - 48 \binom{n}{2} - 20 \binom{n}{1} - 6 \binom{n}{0}$$

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This talk is based on joint work with Lapo Cioni and Luca Ferrari

When sorting permutations, various stack-based structures have been extensively studied since Knuth's introduction of the stacksorting procedure [6]. We present a sorting device which augments a pop stack with a bypass operation. A pop stack is a last-in, first-out structure, but unlike a standard stack, a pop operation removes all entries at once rather than just the top entry. Our device adds a bypass operation, which allows entries to move directly from input to output, creating a sorting machine more powerful than the classical Popstacksort algorithm introduced by Avis and Newborn [3].

We first provide a characterization of sortable permutations in terms of forbidden patterns:

Theorem 1. *A permutation π can be sorted using a pop stack with a bypass if and only if $\pi \in Av(231, 4213)$, i.e., π avoids the patterns 231 and 4213.*

The enumeration of this class was previously established by Atkinson [1]:

Theorem 2. *The number of permutations of length n that can be sorted using a pop stack with bypass is given by the odd-indexed Fibonacci number F_{2n-1} .*

We establish a new combinatorial proof of this result by developing a bijection between the sortable permutations and a class of restricted Motzkin paths. Specifically, we map each sortable permutation to a unique Motzkin path satisfying the following conditions:

- The path begins with an Up step and ends with a Down step.
- The path never has a Horizontal step immediately preceded or followed by a Down step.
- When the path has a Down step, that step is immediately followed by as many Down steps as necessary to reach the x-axis.

This bijection allows us to establish the enumeration through standard techniques for counting restricted Motzkin paths.

We further investigate the structure of the preimages under our sorting algorithm. For any permutation σ , the set $PSB^{-1}(\sigma)$ consists of all permutations that sort to σ using our device. We develop an algorithm to compute all preimages of a given permutation, which allows us to prove the following results:

Proposition 3. *The permutations having exactly 0, 1, and 2 preimages under the pop stack with bypass sorting algorithm can be characterized as follows:*

- *Permutations with 0 preimages are exactly those not ending with their maximum.*
- *Permutations with 1 preimage are those ending with their maximum where all left-to-right maxima are consecutive and non-adjacent.*
- *Permutations with 2 preimages have a specific structure where all left-to-right maxima except possibly the first one are consecutive and non-adjacent.*

We determine precisely when the preimage of a principal class under our sorting operation forms a permutation class itself:

Proposition 4. *Let ρ be a permutation. The set $PSB^{-1}(Av(\rho))$ is a permutation class if and only if ρ has one of the following forms:*

- $\rho = n\alpha$ for some $\alpha \in S_{n-1}$.
- $\rho = (n-1)\alpha n$ for some $\alpha \in S_{n-2}$.

In all other cases, $PSB^{-1}(Av(\rho))$ is not a permutation class.

For the cases where the preimage is a class, we are able to provide explicit bases (i.e., minimal sets of forbidden patterns).

We then extend our analysis to consider a device consisting of two pop stacks in parallel with a bypass. Building on work by Atkinson and Sack [4], who studied pop stacks in parallel (without bypass) and proved that the set of sortable permutations for any number k of pop stacks in parallel is characterized by a finite set of forbidden patterns, we prove:

Theorem 5. *The set of permutations sortable by two pop stacks in parallel with a bypass is precisely the permutation class*

$$Av(2341, 25314, 42513, 42531, 45213, 45231, 52314, 642135, 642153).$$

More generally, we prove:

Theorem 6. *There is a finite set of permutations B_k such that a permutation π is sortable by k pop stacks in parallel with a bypass if and only if $\pi \in Av(B_k)$.*

Following the approach originally applied to classical pop stacks in [7], we determine that the generating function for the permutations sortable by two pop stacks in parallel with a bypass is:

$$\frac{(1-x)(1-2x)(1-4x)}{1-8x+20x^2-18x^3+3x^4}. \quad (15)$$

This generating function was computed using the regular insertion encoding approach developed by Vatter [8] and the automation packages by Albert et al. [2] and Bean et al. [5]. We extend this result to prove:

Theorem 7. *For any positive integer k , the set of permutations sortable by k pop stacks in parallel with a bypass has a rational generating function.*

This work opens several directions for future research, including the extension to words (rather than permutations). We also make the following conjecture based on some initial numerical results.

Conjecture 8. *Let a_n be the number of simple permutations of size n that can be sorted by a machine consisting of two pop stacks in parallel with a bypass. Then $a_0 = a_1 = 1$, $a_2 = 2$, $a_n = F_{2n-5} - 1$ if $n \geq 3$ is odd, and $a_n = F_{2n-5}$ if $n > 3$ is even (where F_n is the n -th Fibonacci number).*

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JOINT PACKING DENSITIES AND THE GREAT LIMIT SHAPE

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This talk includes results due to Sergi Elizalde, Anna de Mier, and Marc Noy (jointly), Lara Pudwell, and Miles Jones.

In this talk I'll summarize what is known about the "Great Limit Shape" (as I have been calling it), and about the "Small Limit Shape," which is the analogous construction for layered permutations. I'll consolidate results presented last fall at the Albany AMS meeting with earlier results by Sergi Elizalde and his team, some experimental observations by Lara Pudwell and Miles Jones, and some new results.

The subject is joint packing densities. If $\sigma \in \mathcal{S}_k$ (a "pattern") and $\pi \in \mathcal{S}_n$, then the *packing density* of σ in π is

$$\delta_\sigma(\pi) = (\text{the number occurrences of } \sigma \text{ in } \pi) / \binom{n}{k}.$$

In this talk, the patterns π always have length 3.

Given a set of patterns—say, 123 and 321—we can form their joint packing density, or *packing vector*,

$$(\delta_{123}(\sigma), \delta_{321}(\sigma)) \in \mathbb{R}^2$$

for each $\sigma \in \mathcal{S}_n$. We would like to know what vectors can arise in this way.

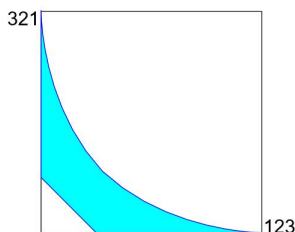
Actually, we only care about the limit for large n . So we define the *limit shape*, $\Pi(123, 321)$, as the set of vectors in $\mathbb{R}^2$ that can occur as the limit of vectors

$$(\delta_{123}(\sigma_i), \delta_{321}(\sigma_i))$$

for a sequence $\{\sigma_1, \sigma_2, \dots\}$ of permutations of increasing size. The limit shape is a compact subset of $\mathbb{R}^2$.

(The limit shapes can also be understood in terms of permutons. A vector is in the limit shape if and only if it can occur as $(\delta_{123}(P), \delta_{321}(P))$ for some permuton P .)

For the patterns 123 and 321, the limit shape is known exactly. It was found by Sergi Elizalde and his collaborators and proved by connecting it to an analogous theorem for graphs.



$\Pi(123, 321)$ (shown as a subset of the unit square $[0, 1]^2$)

We would really like to understand the mother of all limit shapes, the Great Limit Shape:

$$\Pi^* = \Pi(123, 132, 213, 231, 312, 321).$$

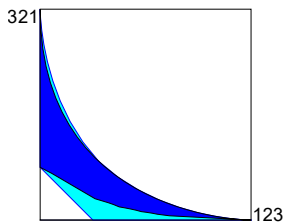
This is a subset of $\mathbb{R}^6$, but because the six packing densities add to 1, the set Π^* is contained in a 5-dimensional subspace of $\mathbb{R}^6$. A full description is still out of reach.

For now, we will have to settle for various projections and cross-sections of Π^* . We have seen $\Pi(123, 321)$, which is a projection of Π^* onto a subspace of dimension 2. Another projection of dimension 2 is $\Pi(132, 321)$, which was partially found by Sergi’s team. Both Lara Pudwell and Miles Jones found evidence to identify the permutons that define the boundaries of $\Pi(132, 321)$, and now we can describe more of the picture, along with a complete description of $\Pi(132, 21)$.

Layered permutations. We can come close to a full solution of the analogous problem for layered permutations. Consider, first, the set $\Pi^L(123, 321)$, which consists of limits of packing vectors

$$(\delta_{123}(\sigma), \delta_{321}(\sigma))$$

where σ is limited to layered permutations. Some of the vectors that occur for general permutations are impossible for layered permutations.

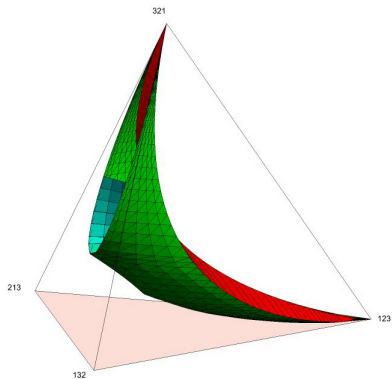
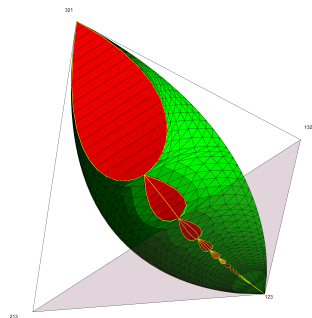


$\Pi^L(123, 321)$ in dark color,
in front of $\Pi(123, 321)$ in
light color

The full limiting shape for layered permutations (the “Small Limit Shape”) consists of limits of vectors of the form

$$(\delta_{123}(\sigma), \delta_{132}(\sigma), \delta_{213}(\sigma), \delta_{321}(\sigma))$$

for layered permutations σ . (We omit the terms for 213 and 312 because these patterns do not occur in layered permutations.) Because entries add to 1, these vectors form a 3-dimensional subset of $\mathbb{R}^4$. It is a cross-section of the Great Limit Shape. Here are two images of the Small Limit Set.



In this talk, we present an ongoing work aiming at systematically solving the enumeration of inversion sequences (and ascent sequences, and restricted growth functions) avoiding 021 and any finite set of patterns.

Context

In [2], an algorithm was developed to automatically find succession rules for generating trees associated with some classes of pattern-avoiding inversion sequences. In particular, the authors used this method to compute the generating function of inversion sequences avoiding the pattern 021 and any pattern of length four or five [4, 5]. Their findings lead to some conjectures about inversion sequences avoiding 021 and another pattern; notably, they appear to always have an algebraic generating function.

Summary of results

We confirm the above conjecture, proving that inversion sequences avoiding 021 and any finite set of patterns have an algebraic generating function. We show that inversion sequences avoiding 021 and any finite set of patterns are in bijection with a language of pattern-avoiding coloured Dyck paths, which is recognized by a deterministic counter automaton. Classical results from formal language theory imply that such a language always has an algebraic generating function. Our algebraicity results also apply to ascent sequences and restricted growth functions avoiding 021 and other patterns.

Inversion sequences avoiding 021, and lattice paths

An *inversion sequence* of length n is a sequence of integers $\sigma = (\sigma_1, \dots, \sigma_n)$ such that $\sigma_i \in \{0, \dots, i-1\}$ for all $i \in \{1, \dots, n\}$. We also consider two subsets of inversion sequences whose values are bounded by different statistics, namely *ascent sequences* and *restricted growth functions*.

An inversion sequence avoids the pattern 10 if and only if it is nondecreasing. There is a simple bijection between 10-avoiding inversion sequences of length n and Dyck paths of length $2n$, see Figure 20 (left).

An inversion sequence avoids the pattern 021 if and only if its positive entries are nondecreasing. Indeed, since the sequence always begins with a 0, any pair of positive entries in decreasing order would complete an occurrence of 021. The bijection between 10-avoiding inversion sequences and Dyck paths can be extended to a bijection between 021-avoiding inversion sequences and a family of coloured Dyck paths.

This bijection first appeared in [3]. See Figure 20 (right) for an illustration.

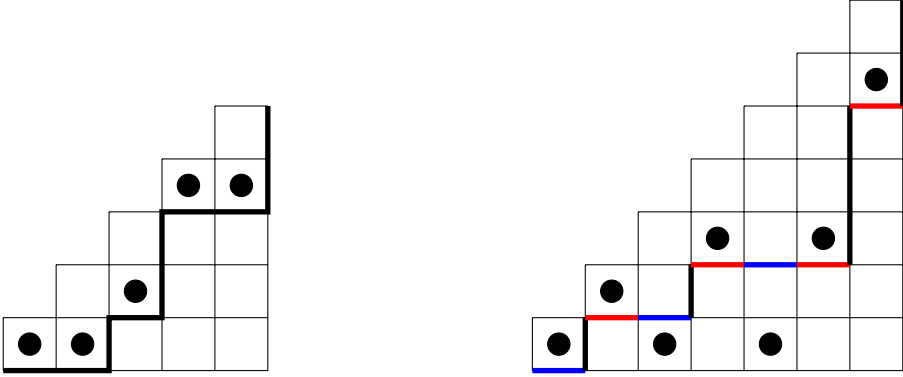


Figure 20: On the left, the 10-avoiding inversion sequence $(0,0,1,3,3)$, and the corresponding Dyck path. On the right, the 021-avoiding inversion sequence $(0,1,0,2,0,2,5)$, and the corresponding coloured Dyck path.

Inversion sequences avoiding 021, and context-free languages

The coloured Dyck paths in bijection with 021-avoiding inversion sequences can be seen as words over the alphabet $\{d, p, z\}$, where each letter z marks a zero entry of the inversion sequence, p marks a positive entry, and d marks a difference between a left-to-right maximum and the previous one (additionally, a suffix of letters d is placed so that the length of the word is exactly twice that of the inversion sequence). For instance, the inversion sequence $(0,1,0,2,0,2,5)$ of Figure 20 is encoded by the word $zdpzdpzpd d d p d d$. These words form a context-free language, which is recognized by the deterministic counter automaton of Figure 21.

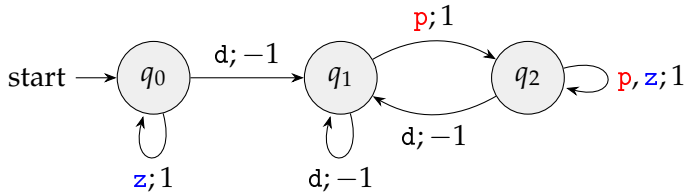


Figure 21: Counter automaton recognizing a language in bijection with 021-avoiding inversion sequences. This automaton accepts a word whenever the counter is 0, and transitions of the form $\ell; -1$ are not allowed when the counter is 0.

The encoding of inversion sequences as coloured Dyck paths is compatible with pattern avoidance in the following sense: a 021-avoiding inversion sequence σ avoids a pattern τ if and only if the Dyck path corresponding to σ avoids one or two patterns associated with τ (they roughly correspond to the encoding of τ as a coloured Dyck path). The previous statement relies on an ad hoc definition of pattern avoidance for words over the alphabet $\{d, p, z\}$, which satisfies a nice property: the set of words

avoiding a pattern is always a rational language.

By classical closure properties of formal languages, it follows that the set of inversion sequences avoiding 021 and any finite set of patterns is in bijection with an unambiguous context-free language. Hence their generating function is algebraic, by the Chomsky–Schützenberger enumeration theorem [1].

The same approach also applies to 021-avoiding ascent sequences, and restricted growth functions. For 021-avoiding restricted growth functions, only rational languages are involved, so the generating functions are even rational.

In addition, with a different encoding of inversion sequences, we show that for any finite set of patterns P which contains 021 and a strictly increasing pattern, the class of P -avoiding inversion (or ascent) sequences has a rational generating function.

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EVERY PATTERN-AVOIDING CLASS OF CATALAN WORDS HAS A RATIONAL GENERATING FUNCTION

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There has been much recent research on pattern-avoiding Catalan words. A *Catalan word* is a word $w_1 \dots w_n$ in alphabet $\mathbb{N}$ such that $w_1 = 0$ and $w_{i+1} \leq w_i + 1$ for all i . These are counted by the Catalan numbers, since the function that maps $w_1 \dots w_n$ to the Dyck path with i th up-step at height w_i is a bijection.

A few different kinds of patterns have been studied in the context of Catalan words. Our focus is on what one might call “classical” patterns, as opposed to consecutive or vincular patterns. Under this definition, if $u = u_1 \dots u_k$ and $v = v_1 \dots v_l$ are words in alphabet $\mathbb{N}$, then we say u is *contained* in v if there exist $i_1, \dots, i_k$ such that $\text{st}(u) = \text{st}(v_{i_1} v_{i_2} \dots v_{i_k})$, where $\text{st}(w)$ denotes the standardization of a word w . That is, u is contained in v if v has a (not necessarily consecutive) subsequence with the same relative order as u . This makes the set of Catalan words into a graded poset, just like the graded poset of permutations under pattern containment, whose down-sets are the classes of Catalan words avoiding a given set of Catalan words. Recent research on Catalan words using this definition of pattern containment [3, 2, 4] focuses on systematic enumeration of the Catalan words that avoid one pattern of size 3, two patterns of size 3, or one pattern of size 4, refined by descent number.

Baril, Mansour, Ramírez, and Shattuck [4] conjecture that every pattern-avoiding class of Catalan words has a rational generating function. Albert and Atkinson [1] proved that the permutation class avoiding 132 contains no infinite antichains and that every proper subclass of it has a rational generating function. Unfortunately, the natural bijection between 132-avoiding permutations and Dyck paths induces a different poset structure from the one on Catalan words described above. Nevertheless, we prove that the set of Catalan words has no infinite antichains, and we prove the conjecture of [4].

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Vamvakari [7], had introduced q -order statistics (that, is q -analogues of the classical order statistics, for $0 < q < 1$) which arisen from dependent and not identically distributed q -continuous random variables and had studied their distributional properties. She had studied the distribution functions and the q -density functions of the relative q -ordered random variables. In this work, we expand the definition of q -ordered random variables to also include discrete q -distributed random variables. We then show how permutation patterns and permutation statistics arise in the study of the distributional properties of q -ordered random variables.

Brief Introduction

The theory of order statistics and their associated properties has been the subject of extensive investigation over the past several decades. A substantial body of literature is devoted to the analysis of order statistics derived from independent and identically distributed (i.i.d.) random variables. More recently, increasing attention has been directed toward the study of order statistics arising from independent or dependent, but not necessarily identically distributed, random variables—an area that remains of significant theoretical and practical interest. Notably, the study of order statistics shares connections with the theory of permutation patterns, particularly in the context of the relative ordering of sample elements. Understanding the structure and frequency of specific permutations induced by ordered samples contributes to both probabilistic and combinatorial analyses, thereby offering a fruitful interplay between the two fields.

In the field of discrete q -distributions, Charalambides [6, p.167] has presented the order statistics arising from independent and identically distributed random variables, with common distribution a discrete q -uniform distribution. Charalambides [4, 5] also has studied the distributions of the record statistics in q -factorially increasing populations.

Vamvakari [7], had introduced q -order statistics, as q -analogues of the classical order statistics, for $0 < q < 1$, which arisen from dependent and not identically distributed q -continuous random variables and had studied their distributional properties. She had studied the distribution functions and the q -density functions of the relative q -ordered random variables. She had focused focus on q -ordered variables arising from dependent and not identically q -uniformly distributed random variables and she derived their q -distributions, including q -power law, q -beta and q -Dirichlet distributions. The motivation for introducing q -order statistics was given by studying the properties of the waiting times of the Heine process (that, is a q -analogue of the classical Poisson process).

In this work, we expand the definition of q -ordered random variables to also include discrete q -distributed random variables. We then highlight the role of permutation

patterns and permutation statistics in the study of distributional properties of the considered q -ordered random variables.

Main Results

Let a ν -variate random variable $\mathcal{X} = (X_1, \dots, X_\nu)$ be defined in a sample space Ω . Then for the values $x_1 = X_1(\omega), \dots, x_\nu = X_\nu(\omega)$, $\omega \in \Omega$ there is a permutation $(i_1, \dots, i_\nu)$ of the ν indices $\{1, \dots, \nu\}$, such that $x_{i_1} \leq \dots \leq x_{i_{\nu-1}} \leq x_{i_\nu}$. The k -th ordered random variable is denoted by $X_{(k)}$ and defined by

$$X_{(k)}(\omega) = x_{(k)}, \quad \omega \in \Omega,$$

where $x_{(k)} = x_{i_k}$, $k = 1, \dots, \nu$. In particular, for $k = 1$ this gives $X_{(1)} = \min\{X_1, \dots, X_\nu\}$ and, for $k = \nu$ this gives $X_{(\nu)} = \max\{X_1, \dots, X_\nu\}$. Generally, the following inequalities hold:

$$X_{(1)} \leq \dots \leq X_{(\nu)}.$$

Building on the notion of q -integral introduced by Thomae in 1869, Vamvakari [7], defined q -order statistics-i.e., q -analogues of the classical order statistics- for $0 < q < 1$, which arisen from dependent and not identically distributed q -continuous random variables and had studied their distributional properties. In this work, we expand the definition of q -ordered random variables to also include discrete q -distributed random variables, as follows:

Definition 1 (q -ordered). Let $\mathcal{Y} = (Y_1, \dots, Y_\nu)$ be a ν -variate random variable and $Y_{(k)}$, $1 \leq k \leq \nu$, be the corresponding k -th ordered random variables. Assume that $Y_{(k)}$, $1 \leq k \leq \nu$, satisfy the inequalities

$$0 \leq Y_{(1)} < qY_{(2)} < Y_{(2)} < \dots < Y_{(\nu-1)} < qY_{(\nu)} < Y_{(\nu)} \leq \beta, \beta > 0. \quad (16)$$

Then, $Y_{(k)}$ (for any k such that $1 \leq k \leq \nu$) is called the k -th q -ordered random variables.

We then show how permutation patterns and permutation statistics arise in the study of the distributional properties of q -ordered random variables. Analytically, we present generalized versions of theorems originally established by the author in 2023, highlighting the role of permutation patterns and permutations statistics in our study.

Theorem 2. Let $Y_1, \dots, Y_\nu$ be dependent random variables, where $Y_{(i)}$, $i = 1, 2, \dots, \nu$, satisfy inequalities (16). Then, the distribution function of the k -th q -ordered random variable $Y_{(k)}$, $1 \leq k \leq \nu$, is given for $y \in [0, \beta]$ by

$$\begin{aligned} F_{Y_{(k)}}(y) &= P(Y_{(k)} \leq y) \\ &= \sum_{r=k}^{\nu} \sum_{1 \leq i_1 < \dots < i_r \leq \nu} \prod_{j=1}^r F_{Y_{i_j}}(q^{j-1}y) \prod_{m=r+1}^{\nu} \left(1 - F_{Y_{i_m}}(q^{i_m - (m-r)}y)\right), \end{aligned} \quad (17)$$

where the inner summation is over all r -combinations $\{i_1, \dots, i_r\}$ of the set $\{1, \dots, \nu\}$ and $F_{Y_i}(y) = P(Y_i \leq y)$ is the distribution function of each Y_i , $i = 1, 2, \dots, \nu$.

Theorem 3. Let $Y_1, \dots, Y_\nu$ be dependent random variables, where $Y_{(i)}, i = 1, 2, \dots, \nu$, satisfy inequalities (16). Then, the joint distribution function of the q -ordered random variables $Y_{(k)}$ and $Y_{(r)}$ for $1 \leq k < r \leq \nu$, is given for $y, z \in [0, \beta]$ by

$$\begin{aligned} F_{Y_{(k)}, Y_{(r)}}(y, z) &= P(Y_{(k)} \leq y, Y_{(r)} \leq z) \\ &= \sum_{j=r}^{\nu} \sum_{s=k}^j \sum_{n_1=1}^s F_{Y_{i_{n_1}}}(q^{n_1-1}y) \prod_{n_2=s+1}^j (F_{Y_{i_{n_2}}}(q^{n_2-s-1}z) - F_{Y_{i_{n_2}}}(y)) \\ &\quad \cdot \prod_{n_3=j+1}^{\nu} \left(1 - F_{Y_{i_{n_3}}}(q^{i_{n_3}-(n_3-j)}z)\right), y < q^{r-k}z, 1 \leq k < r \leq \nu, \end{aligned} \quad (18)$$

where the inner summation is over all pairwise disjoint subsets $\{i_1, \dots, i_s\}$ and $\{i_{s+1}, \dots, i_j\}$ of the set $\{1, \dots, \nu\}$ with $1 \leq i_1 < \dots < i_s \leq \nu$ and $1 \leq i_{s+1} < i_{s+2} < \dots < i_j \leq \nu$, and $F_{Y_i}(y) = P(Y_i \leq y)$ is the distribution function of each $Y_i, i = 1, 2, \dots, \nu$.

Remark 4. The above theorems yield, among other results, distributional properties of order statistics arising from dependent and non-identically distributed random variables, specifically those following q -uniform or discrete q -uniform distributions.

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This talk is based on joint work with Bogdan Alecu, Robert Ferguson, Mamadou Moustapha Kanté, Vadim Lozin, Sean Mandrick, and Viktor Zamaraev

Geometric grid classes of permutations [1], are defined using matrices whose entries indicate monotone segments in specified grid cells. Given a matrix, its geometric grid class includes all permutations whose plots can be “drawn on” the corresponding grid figure.

Lettericity, introduced by Petkovšek [2], is a graph parameter defined via representations known as letter graphs. A letter graph is constructed from a word over a finite alphabet together with a decoder—a specified set of letter pairs determining adjacency. Vertices correspond to letters of the word, and edges between vertices are dictated by the decoder. The lettericity of a graph is the smallest alphabet size needed for such a representation.

Though initially appearing unrelated, geometric griddability and lettericity share deep structural similarities. This talk presents results from Alecu, Ferguson, Kanté, Lozin, Vatter, and Zamaraev [3] demonstrating that a permutation class is geometrically griddable if and only if the associated graph class has bounded lettericity. The discussion will also include results by Mandrick and Vatter [4] on the typical lettericity of graphs: almost all graphs on n vertices have lettericity between $n - \frac{1}{2} \log_2 n$ and $n - (2 \log_2 n + 2 \log_2 \log_2 n)$. Finally, the talk will conclude by contrasting this result with the analogous problem for permutations.

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This talk is based on joint work with Svante Linusson

In 2012, Claesson, Jelínek and Steingrímsson tackled the notoriously difficult problem of enumerating the 1324-avoiders, leading to the first interesting upper bound for their Stanley–Wilf limit [2]. They also formulated the *inversion-monotonicity conjecture*, whose proof would improve the upper bound even further. In this work, we prove half of the conjecture. Let $\text{Av}_n^k(\tau)$ denote the set of τ -avoiders of length n with k inversions, and $\text{av}_n^k(\tau) = |\text{Av}_n^k(\tau)|$.

Conjecture 1 (Conjecture 13 in [2]). *For all nonnegative integers n and k ,*

$$\text{av}_n^k(1324) \leq \text{av}_{n+1}^k(1324).$$

Any $\pi \in \mathfrak{S}_n$ with $n \geq \text{inv}(\pi) + 2$ is (sum) decomposable. A decomposable 1324-avoider is necessarily of the form

$$\pi = \pi^{(1)} \oplus 1 \oplus 1 \oplus \dots \oplus 1 \oplus \pi^{(2)},$$

where $\pi^{(1)} \in \text{Av}(132)$ and $\pi^{(2)} \in \text{Av}(213)$; therefore

$$\text{av}_n^k(1324) = \sum_{i=0}^k p(i)p(k-i) =: a(k) \quad \text{for all } n \geq k+2, \quad (19)$$

where $p(k)$ is the number of integer partitions of k . Using the classical estimate $p(k) < \exp(\pi\sqrt{2k/3})$ along with (19), one can show that Conjecture 1 implies the bound

$$L(1324) < \exp(\pi\sqrt{2/3}) \approx 13.001594.$$

This would still today be the best known upper bound for the Stanley–Wilf limit of the 1324-avoiders: the current records are $10.27 \leq L(1324) \leq 13.5$ due to Bevan, Brignall, Elvey Price and Pantone [1]. Convincing estimates by Conway, Guttmann and Zinn-Justin point at the actual value being $L(1324) \approx 1.600 \pm 0.003$ [3].

Since 2012, Conjecture 1 has received much attention. Our main result marks the first bit of progress: the conjecture holds for all k and n such that $n \geq \frac{k+7}{2}$. The proof relies on a structural characterization of the permutations in question in terms of a new notion of *almost-decomposability*, leading to an enumeration.

Theorem 2. *For all nonnegative integers k and $n \geq \frac{k+7}{2}$,*

$$\text{av}_n^k(1324) = a(k) - 4a(k-n+1) - 6 \sum_{i=0}^{k-n} a(i), \quad (20)$$

where $a(k) = \sum_{i=0}^k p(i)p(k-i)$ and $p(k)$ is the number of integer partitions of k . In particular,

$$\text{av}_{n+1}^k(1324) - \text{av}_n^k(1324) = 4a(k-n+1) + 2a(k-n) \geq 0,$$

and this difference has a combinatorial interpretation.

Remark 3. In the language of generating functions, Theorem 2 states that

$$\text{av}_n^k(1324) = [x^k] \left(P(x)^2 - \frac{R_n(x)}{1-x} \right)$$

whenever $n \geq \frac{k+7}{2}$, where $P(x) = \sum_{i \geq 0} p(i)x^i$ is the generating function for the partition numbers and $R_n(x) = 2(2+x)x^{n-1}P(x)^2$. In particular,

$$\text{av}_{n+1}^k(1324) - \text{av}_n^k(1324) = [x^k]R_n(x).$$

This wording is more convenient for the proof.

Observe that Conjecture 1 holds (with equality) for all k and n such that $n \geq k+2$ due to (19). It therefore remained to prove that the conjecture holds when $n \leq k+1$, and after Theorem 2, only the part $n < \frac{k+7}{2}$ remains. Table 9 shows the values of $\text{av}_n^k(1324)$ for $n \leq 20$, $k \leq 25$. The constant part of each column is colored in blue; the sequence $1, 2, 5, 10, 20, \dots$ is given by $a(k)$ from (19). Theorem 2 gives the values contained in the red cells with formula (20). The uncolored cells mark the region for which Conjecture 1 is still open. A larger version of the table is available at <https://akc.is/inv-mono/> courtesy of Anders Claesson.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1																									
2	1	1																								
3	1	2	2	1																						
4	1	2	5	6	5	3	1																			
5	1	2	5	10	16	20	20	15		9	4	1														
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1										
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20	6	1				
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441	1173	866	574	338	174	76
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654	7559	8225	8545	8457	7930	7006
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521	15012	19067	23599	28426	33300	37862
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915	18976	25563	34017	44640	57739	73421
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312	21171	28973	39338	52919	70657	93482
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216	22632	31266	42845	58213	78531	105137
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786	23568	32768	45234	61902	84130	113477
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118	24138	33728	46776	64346	87939	119306
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302	24470	34298	47736	65916	90431	123184
17	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5818	8456	12192	17402	24654	34630	48306	66876	92001	125708
18	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5822	8466	12216	17452	24754	34814	48638	67446	92961	127278
19	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5822	8470	12226	17476	24804	34914	48822	67778	93531	128238
20	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5822	8470	12230	17486	24828	34964	48922	67962	93863	128808

Table 9: The numbers $\text{av}_n^k(1324)$.

Our proof of Theorem 2 uses a structural characterization of 1324-avoiders with few inversions. We say that a permutation $\pi \in \mathfrak{S}_n$ is *almost decomposable* if it is indecomposable, but

$$\max \{ \text{comp}(\pi \setminus e) : e \in \{1, \pi_1, n, \pi_n\} \} \geq 2. \quad (21)$$

Here $\pi \setminus \pi_i$ denotes the reduction of $\pi_1 \dots \pi_{i-1} \pi_{i+1} \dots \pi_n$, and $\text{comp}(\pi)$ is the number of sum components of π .

Theorem 4. *Every indecomposable 1324-avoider π such that $\text{inv}(\pi) \leq 2|\pi| - 7$ is almost decomposable.*

Almost-decomposability allows us to construct an injection $\text{Av}_{n+1}^k(1324) \rightarrow \text{Av}_n^k(1324)$ in a natural way when $n \geq \frac{k+7}{2}$. If $\pi \in \text{Av}(1324)$ is decomposable, it is of the form

$$\pi = \pi^{(1)} \oplus \underbrace{1 \oplus 1 \oplus \dots \oplus 1}_{m \text{ times}} \oplus \pi^{(2)},$$

and we can set

$$f(\pi) = \pi^{(1)} \oplus \underbrace{1 \oplus 1 \oplus \dots \oplus 1}_{m+1 \text{ times}} \oplus \pi^{(2)}.$$

On the other hand, if π is indecomposable but $\pi \setminus \pi_1$ is decomposable, we let $f(\pi)$ be the permutation σ such that $\sigma_1 = \pi_1$ and $\sigma \setminus \sigma_1 = f(\pi \setminus \pi_1)$. This is easily extended to the other cases of (21) with symmetries, and f preserves both 1324-avoidance and inversion number.

Theorem 5. *Let k and n be nonnegative integers such that $n \geq \frac{k+7}{2}$. The map*

$$f : \text{Av}_n^k(1324) \longrightarrow \text{Av}_{n+1}^k(1324)$$

we defined above is injective.

The remainder of the proof of Theorem 2 relies on a careful enumeration of the collection of permutations $\text{Av}_{n+1}^k(1324) \setminus \text{im } f$. Complete proofs of Theorems 2, 4 and 5 are available in our preprint [4]. At least two natural questions arise:

- Can our method be extended to prove a larger part of Conjecture 1?
- Does almost-decomposability have other uses?

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EXHAUSTIVE GENERATION OF PATTERN-AVOIDING s -WORDS

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This talk is based on joint work with Samuel Buick, Madeleine Goertz, Amos Lastmann, Kunal Pal, Helen Qian, Sam Tacheny, Leah Williams, and Yulin Zhai (NSF Grant DMS2241623)

We introduce a simple approach for generating Gray codes of pattern-avoiding s -words (i.e., multiset permutations) and corresponding combinatorial objects. It generalizes plain changes and the recent *Combinatorial Generation via Permutation Language* series.

Introduction *Plain changes* is a swap Gray code for S_n the permutations of $[n] = \{1, \dots, n\}$: e.g., $\underline{123}, \underline{132}, \underline{312}, \underline{321}, \underline{231}, \underline{213}$ where a *swap* moves a large digit past a small digit. It was discovered in the 1600s and is also known as the *Steinhaus-Johnson-Trotter algorithm*. More recently, it has been viewed as a greedy algorithm: “swap the largest value” [6].

A *permutation language* is any $L \subseteq S_n$. A *jump* moves a larger digit past **one or more smaller** digits. It is *minimal* for $w \in L$ if its *distance* d (i.e., number of smaller digits) is minimized to create $w' \in L$. Algorithm J debuted at *Permutation Patterns 2019*: “minimal jump the largest value” [2]. It generates zig-zag languages e.g., $\underline{123}, \underline{132}, \underline{312}, \underline{321}, \underline{213}$ for $\text{Av}_3(231)$. But jumps become limited for s -words (or s -permutations) which have s_i copies of i for $i \in [m]$. For example, $w = 123332$ is a *Stirling s -word* for $s = (1,2,3)$ (i.e., $L = \text{Av}_s(212)$) but any jump in w gives an *invalid* $w' \notin L$. The set of all s -words is S_s . Each $v \in [m]$ is a *value*, each copy of a value in w is a *digit*, and a digit’s *rank* is by increasing value then left-to-right index. For example, the ranks of 123332 are $\underline{124563}$.

We consider “bumps” which move a **run** of some larger value. Algorithm B generates many s -word languages (i.e., subsets of S_s). Several applications are listed below.

- (a) Gray codes for S_s (e.g., $\underline{1122}, \underline{1221}, \underline{1212}, \underline{2112}, \underline{2121}, \underline{2211}$ for $s = (2,2)$) using transpositions.
- (b) *Stirling changes* generalizes plain changes to *Stirling s -words* $\text{Av}_s(212)$ using transpositions.
- (c) Bump Gray code for regular words counted by k -Catalan numbers $\text{Av}_m^{k-1}(132,121)$ [3, 7].

Our Gray codes lead to efficient algorithms and have various applications (see Figures). For example, we generate (b) *looplessly* (i.e., worst-case $O(1)$ -time per word) in Algorithm 1. In addition, (b) leads to a ± 1 tree-inversion Gray code for s -increasing trees that proves Theorem 1. Finally, (c) gives Gray codes for various k -Catalan objects.

Theorem 1. *Every s -permutahedron has a Hamilton path. (See [1] and its decreasing trees.)*

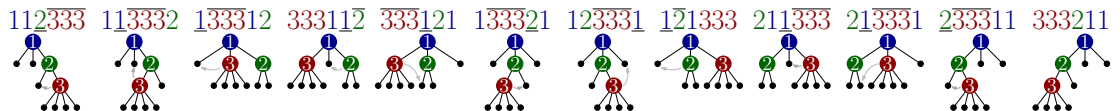


Figure 1: Stirling changes for Stirling s -words with $s = (2,1,3)$ (i.e., permutations of $\{1,1,2,3,3,3\}$ avoiding 212) as generated by Algorithm B with its corresponding s -increasing tree Gray code.

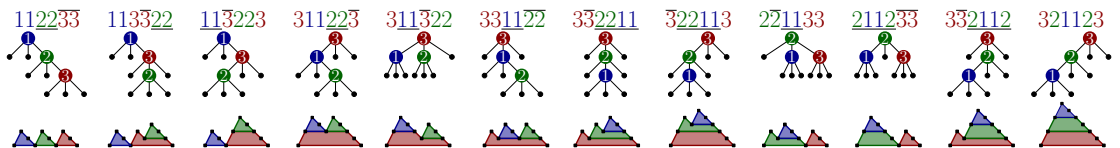


Figure 2: Pattern-avoiding s -words $\text{Av}_s(132,121)$ for $s = (2,2,2)$ with Gray codes for 3-Catalan objects. The ternary trees differ by moves preserving inorder (i.e., visit self before last child).

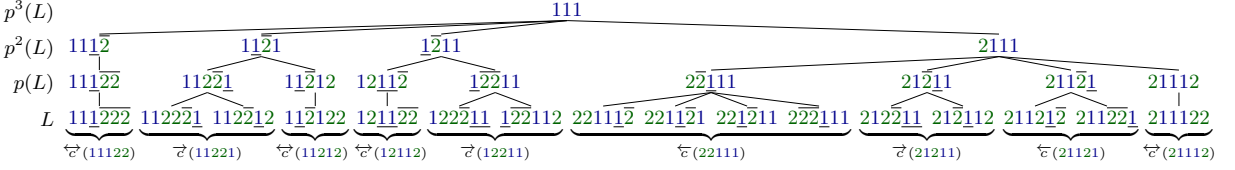


Figure 3: Algorithm B 's Gray code for $L = \text{Av}_{3,3}(12121)$ and its ancestor languages. Successive children jump the rightmost largest value while a bump changes the last child of one parent to the first child of the next parent. Note that the rightmost largest value may join these bumps. For example, the jump $22\bar{1}11 = 21211$ in $p(L)$ widens to the bump $222\bar{1}11 = 212211$ in L .

Algorithm B for Bumps Let $w = w_1w_2 \cdots w_n$ be an s -word with $s = (s_1, s_2, \dots, s_m)$. The *right-maximal run* (or *right-run*) at index i is $w_iw_{i+1} \cdots w_j$ with $w_i = \cdots = w_j$ and $j = n$ or $w_j \neq w_{j+1}$. A right-run is also determined by w_i 's rank. A *right-bump* moves a right-run to the right past some smaller digits. Left notions are similar. A bump's *rank* r , *width* w , and *distance* d are the rank of w_i , # of larger digits, and # of smaller digits. For example, $11211\bar{3}3311$ creates $1\bar{3}31211\bar{3}11$ by a left-bump of $r = 9$, $w = 2$, and $d = 4$. Bumps can be jumps ($w = 1$), transpositions ($d = 1$), or swaps ($w = d = 1$).

Let L be an s -word language. Suppose a bump at rank r changes $w \in L$ to w' . The bump is *minimal* for its rank and direction if its distance is positive and minimized with $w' \in L$. Minimal bumps on $w \in L$ are uniquely determined by rank and direction. Our generalization of Algorithm J [2] replaces values with ranks, and prefers rightwards.

Algorithm B (Greedy Bumps). This algorithm attempts to generate an s -word language L with $s = (s_1, s_2, \dots, s_m)$ starting from a given or default initial word $w \in L$.
B1. [Initialize] Visit the given initial word w (or by default visit $w = 1^{s_1}2^{s_2} \cdots m^{s_m}$).
B2. [Greedy] Let w be the most recent word visited. Visit a new word by applying to w a minimal bump prioritized by largest rank then rightward over leftward. Halt if no such bump exists. Otherwise, repeat B2.

A jump is *maximum* if it uses the longest distance from an index in a direction. A *zig-zag language* is an s -word language closed under maximum jumps (c.f., [2]). This includes $\text{Av}_s(\alpha)$ when α 's largest values are *internal* (i.e., not first or last) and *isolated* (i.e., not consecutive). Zig-zag languages are closed under intersection (and union) and note that the *peakless s-words* $\text{Av}_s(132, 231, 121)$ are a subset of every zig-zag language.

Theorem 2.¹ If L is zig-zag language of s -words for $s = (s_1, s_2, \dots, s_m)$, then Algorithm B generates a bump Gray code for L starting from the non-decreasing word $w = 1^{s_1}2^{s_2} \cdots m^{s_m} \in L$.
Sketch. Consider the end of an inductive argument on $n = \sum s$. The parent $p(s)$ of s is $(s_1, s_2, \dots, s_m - 1)$ if $s_m > 1$ or $(s_1, s_2, \dots, s_{m-1})$ if $s_m = 1$. Similarly, $p(w)$ removes the rightmost m from $w \in L$, and $p(L) = \{p(w) \mid w \in L\}$. The children of $w' \in p(L)$ are $c(w') = \{w \in L \mid p(w) = w'\}$. Since L is a zig-zag language, $c(w')$ has s -words where the rightmost m is at (a) index n , and (b) index 1 if $s_m = 1$ or beside the rightmost m in w' if $s_m > 1$; these extremes are equal when w' ends in m . Let $\vec{c}(w')$ list $c(w')$ in lexicographic order (i.e., the rightmost m jumps left-to-right) and $\overleftarrow{c}(w')$ in reverse; use $\overleftarrow{\overleftarrow{c}}(w')$ when w' has one child. Note that $p(L)$ is a zig-zag language, so Algorithm B generates w' . We claim that Algorithm B generates $\vec{c}(w')$ or $\overleftarrow{c}(w')$ (or $\overleftarrow{\overleftarrow{c}}(w')$) when run on L . In other words, it unfurls a *local recursion*. In particular, a bump transforms the last child of each parent to the first child of the next parent. See Figure 3. $\square$

¹The full paper generalizes this to *zig/zag languages* (rather than zig-zag languages) including $\text{Av}_s(212)$.

Algorithm 1 Loopless generation of Stirling s -words $\text{Av}_s(212)$ in Stirling changes order.

```

def Stirling( $s_1, s_2, \dots, s_m$ )  $\triangleright$  1-based indexing
1:  $t_1, t_2, t_3, \dots, t_m \leftarrow 0, s_1, s_1 + s_2, \dots, s_1 + \dots + s_{m-1}$ 
2:  $\text{perm} \leftarrow 1^{s_1} 2^{s_2} \dots m^{s_m}$   $\triangleright$   $s$ -word ( $s$ -perm)
3:  $\text{left} \leftarrow [t_1 + 1, \dots, t_{m-1} + 1]$   $\triangleright$  left indices  $1, \dots, m$ 
4:  $\text{inv} \leftarrow 0^m$   $\triangleright$  # of smaller digits right of  $1, 2, \dots, m$ 
5:  $\text{fs} \leftarrow 12 \dots m$   $\triangleright$  focus pointers for  $1, \dots, m$  [4, 5]
6:  $\text{dirs} \leftarrow -1^m$   $\triangleright$  bump directions for  $1, \dots, m$ 
7:  $v \leftarrow \text{fs}[m]$   $\triangleright$  larger value in next bump
8: while  $v > 1$  do  $\triangleright$  1 is never the larger value
9:    $d \leftarrow \text{dirs}[v]$   $\triangleright$  direction of  $v$ 
10:  if  $d = 1$  then  $\triangleright$  right-bump  $v$ 's run?
11:     $i \leftarrow \text{left}[v]$   $\triangleright$  first index of  $v$ 
12:     $j \leftarrow \text{left}[v] + s[v]$   $\triangleright$  index after  $v$ 's run
13:  else  $\triangleright$  left-bump  $v$ 's run
14:     $i \leftarrow \text{left}[v] + s[v] - 1$   $\triangleright$  last index of  $v$ 
15:     $j \leftarrow \text{left}[v] - 1$   $\triangleright$  index before  $v$ 's run
16:   $u \leftarrow \text{perm}[j]$   $\triangleright$  smaller digit to bump
17:  visit  $\text{perm}$   $\triangleright$  next  $s$ -word in Gray code
18:   $\text{perm}[i] \leftarrow u$   $\triangleright$  apply bump to the  $s$ -word
19:   $\text{perm}[j] \leftarrow v$   $\triangleright$  (as a transposition)
20:   $\text{left}[v] \leftarrow \text{left}[v] + d$   $\triangleright$  leftmost  $v$  moved
21:  if  $\text{left}[u] = j$  then  $\triangleright$  leftmost  $u$  moved?
22:     $\text{left}[u] \leftarrow \text{left}[u] - d \cdot s_v$   $\triangleright$   $u$  passed all  $v$ 
23:   $\text{inv}[v] \leftarrow \text{inv}[v] - d$   $\triangleright$   $v$ 's run passed one  $u$ 
24:  if  $\text{inv}[v] = 0$  or  $\text{inv}[v] = t_v$  then  $\triangleright$   $v$  limit?
25:     $\text{dirs}[v] \leftarrow -d$   $\triangleright$  change  $v$ 's direction
26:     $\text{fs}[v] \leftarrow \text{fs}[v-1]$   $\triangleright$  inherit focus of  $v-1$ 
27:     $\text{fs}[v-1] \leftarrow v-1$   $\triangleright$  reset focus of  $v-1$ 
28:   $v \leftarrow \text{fs}[m]$   $\triangleright$  larger value in next bump
29:   $\text{fs}[m] \leftarrow m$   $\triangleright$  reset focus of  $m$ 
30: visit  $\text{perm}$   $\triangleright$  last  $s$ -word in Gray code

```

perm	v	u	i	j	left	inv	fs	dirs
112333	3	2	6	3	134	000	123	---
113332	3	1	5	2	163	001	123	---
133312	3	1	4	1	162	002	123	---
333112	2	1	6	5	461	003	123	--+
333121	3	1	1	4	451	013	123	--+
133321	3	2	2	5	152	012	123	--+
123331	3	1	3	6	123	011	123	--+
121333	2	1	2	1	124	010	123	---
211333	3	1	6	3	214	020	113	+-
213331	3	1	5	2	213	021	113	+-
233311	3	2	4	1	512	022	113	+-
333211	1				541	023	123	++

Variable trace at **visit** for $s = (2, 1, 3)$.

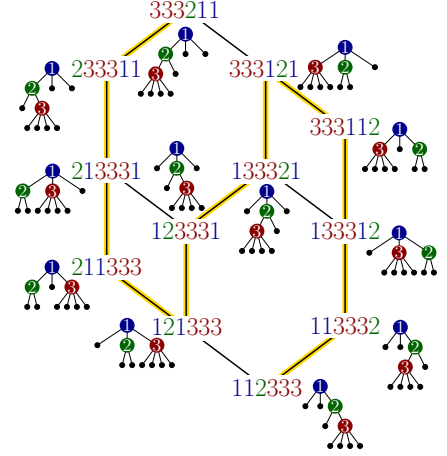


Figure 4: s -permutahedron Hamilton path for $s = (2, 1, 3)$ (or $s = (3, 1, 2)$ [1]).

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