

Inversion sequences avoiding 021 and other patterns

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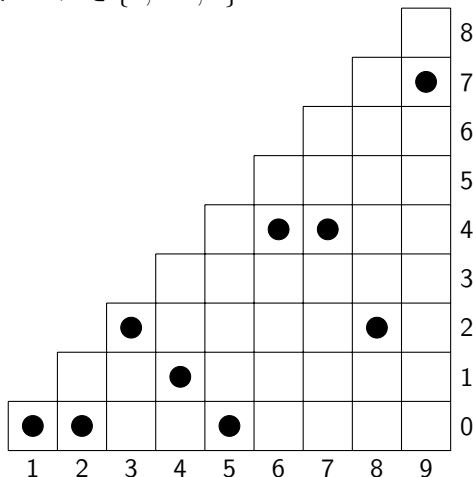
Inversion sequences

An **inversion sequence** is a finite sequence of integers $(\sigma_i)_{i \in \{1, \dots, n\}}$ such that

$$0 \leq \sigma_i < i \quad \forall i \in \{1, \dots, n\}.$$

Example:

$(0, 0, 2, 1, 0, 4, 4, 2, 7) :$



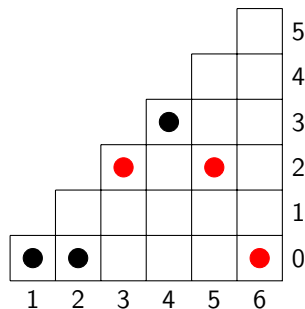
Patterns

A **pattern** is a finite sequence of integers.

A sequence **contains** a pattern if it has a subsequence whose **values** appear in the **same order** as in the pattern.

Example:

The inversion sequence $(0,0,2,3,2,0)$ contains the pattern **110**, and avoids 100.



$\mathcal{I}(\rho)$ is the set of inversion sequences which **avoid** the pattern ρ .

Examples:

- $\mathcal{I}(01) = \{(0), (0,0), (0,0,0), (0,0,0,0), (0,0,0,0,0), \dots\}$
- $\mathcal{I}(00) = \{(0), (0,1), (0,1,2), (0,1,2,3), (0,1,2,3,4), \dots\}$

Pattern-avoiding inversion sequences started with **Carla Savage**'s talk at Permutation Patterns 2014 (Johnson City, Tennessee). Then in two articles, by

- **Mansour** and **Shattuck**,
- **Corteel**, **Martinez**, **Savage**, and **Weselcouch**.

Recently, **Kotsireas**, **Mansour**, and **Yıldırım** used an **algorithmic** approach to **systematically solve** their enumeration, in some cases.

They showed that $\mathcal{I}(021, \rho)$ has an **algebraic** generating function for any pattern ρ of **length ≤ 5** .

In this talk, we show the following.

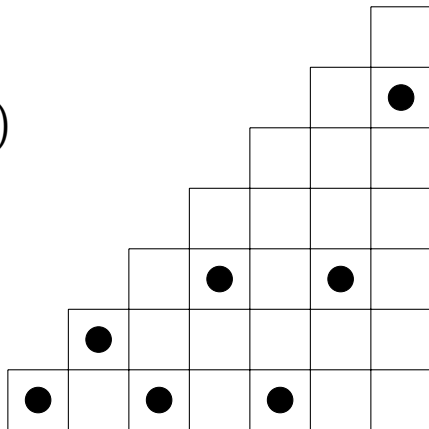
Theorem

$\mathcal{I}(P)$ has an algebraic generating function for any finite set of patterns P that contains 021.

Inversion sequences avoiding 021

An inversion sequence avoids 021 if its **nonzero** entries are **nondecreasing**.
Lin and **Kim** showed that these sequences are in **bijection** with some **colored Dyck paths**.

Example: $(0, 1, 0, 2, 0, 2, 5)$

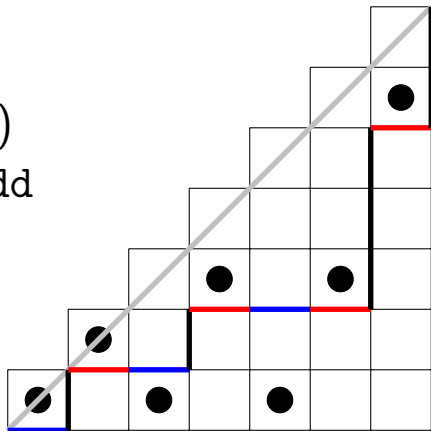


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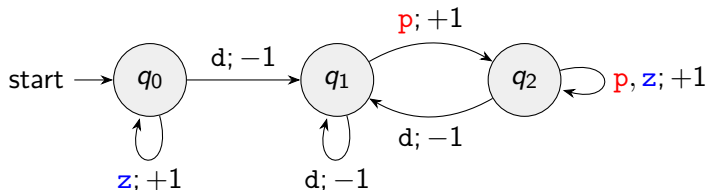
Example: $(0, 1, 0, 2, 0, 2, 5)$

$\mapsto$ **z****d****p****z****d****p****z****p****d****d****d****p****d****d**



Counter automaton for $\mathcal{I}(021)$

This gives a **bijection** between $\mathcal{I}(021)$ and a language over the alphabet $\{d, p, z\}$ that is recognized by a **counter automaton**.



The counter is **initialized at 0** and cannot take negative values. A word is **recognized** whenever the counter is 0.

Avoiding 021 and other patterns

We now consider sequences in $\mathcal{I}(021)$ avoiding some set of patterns P . Their study can be **restricted** to the case $P \subset \mathcal{I}(021)$ **without loss of generality**.

Proposition

For any set of patterns P , there exists a set of patterns P' such that

- $P' \subset \mathcal{I}(021)$,
- *a sequence in $\mathcal{I}(021)$ avoids P if and only if it avoids P' .*

Proof: for each pattern $\rho \in P$,

- if ρ contains 021, we may **remove it** from the set,
- if ρ avoids 021 and **begins with 0**, then $\rho \in \mathcal{I}(021)$,
- if ρ **does not begin with 0**, then $\mathcal{I}(021, \rho) = \mathcal{I}(021, 0 \cdot \rho)$.

Ternary pattern avoidance

The **patterns** are also in $\mathcal{I}(021)$, so they can be **encoded** by **ternary words** over the alphabet $\{d, p, z\}$ as before.

This leads us to define a notion of **ternary patterns** on words, which roughly corresponds to patterns on inversion sequences.

Definition

Let ω and τ be two words over the alphabet $\{d, p, z\}$. We say that ω **contains the ternary pattern** τ if

- the word consisting of the letters **p** and **z** of τ is a **subword** of ω , and
- there is an occurrence of this subword in which two letters **p** are **separated** by a letter **d** in ω if and only if the same holds in ρ .

Examples:

The word $\bar{z}d\bar{p}zdd\bar{p}d \leftrightarrow 0103$ contains the ternary pattern $zdpdpd \leftrightarrow 012$.

The word $\bar{z}d\bar{p}\bar{z}pddd \leftrightarrow 0101$ contains the ternary pattern $zzdpdd \leftrightarrow 001$.

The word $\bar{z}d\bar{p}d\bar{p}zdd \leftrightarrow 0120$ does **not** contain the pattern $zdpdpd \leftrightarrow 011$.

From integer patterns to ternary patterns

Properties:

- A sequence $\sigma \in \mathcal{I}(021)$ contains an occurrence of a pattern ρ which uses a 0 if and only if the ternary word associated with σ contains the ternary pattern associated with ρ .
- If ρ contains a descent, then any occurrence of ρ in σ must use a 0. In this case, both notions of patterns are the same.

If all zeros of ρ are in a prefix, things get more difficult : for instance, $01124 \leftrightarrow \text{zdppdpdd}$ contains the pattern $0012 \leftrightarrow \text{zzdpdpdd}$.

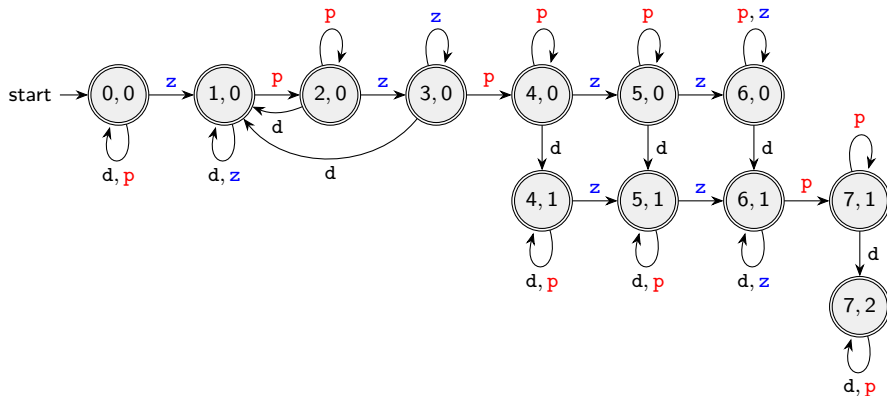
Notice that containing 0012 is the same as containing an occurrence of either 0012 or 01123 which uses a 0.

Avoiding 0012 means avoiding the ternary patterns encoding both patterns: zzdpdpdd and zdppdpdd .

More generally, a pattern ρ of this form is turned into a pair $\{\rho, 0 \cdot (\rho + 1)\}$, resulting in two ternary patterns to avoid.

Automaton for the avoidance of a ternary pattern

The automaton below recognizes words over the alphabet $\{d, p, z\}$ avoiding the (shortened) ternary pattern $zpzpzzdpz \leftrightarrow 01010020$.



Theorem

For any finite set of patterns P , the class $\mathcal{I}(\{021\} \cup P)$ has an algebraic generating function.

Proof: $\mathcal{I}(\{021\} \cup P)$ is in bijection with the **intersection** of languages recognized by

- a **counter automaton**, for $\mathcal{I}(021)$,
- one or several **finite automata**, for pattern-avoidance constraints.

These words form a **context-free language**, recognized by a (deterministic) counter automaton that is the **product** of these automata.

By the **Chomsky–Schützenberger** enumeration theorem, this implies their **generating function** is **algebraic**.

Ascent sequences, restricted growth functions

We now extend the previous result to other types of sequences.

The set of **ascent sequences** is

$$\mathcal{A} = \{\sigma \in \mathcal{I} : \sigma_i \leq 1 + |\{j < i - 1 : \sigma_j < \sigma_{j+1}\}| \quad \forall i\}.$$

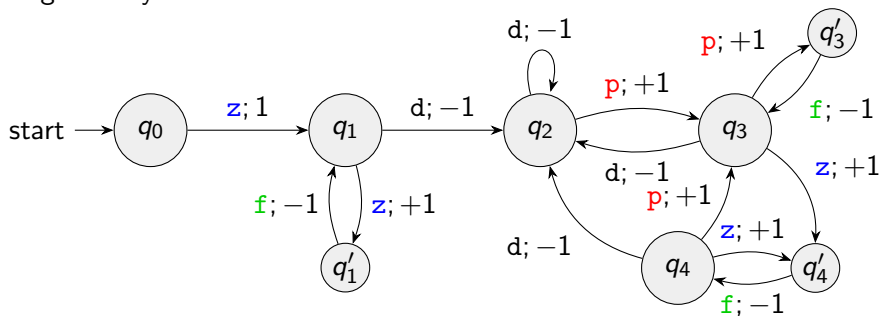
These sequences were introduced in 2010 by **Bousquet-Mélou**, **Claesson**, **Dukes**, and **Kitaev**. Since then, pattern-avoidance has also been studied for these sequences in various articles.

The set of **restricted growth functions** is

$$\mathcal{R} = \{\sigma \in \mathcal{I} : \sigma_i \leq 1 + \max((\sigma_j)_{j < i}) \quad \forall i\}.$$

Ascent sequences avoiding 021

$\mathcal{A}(021)$ is in bijection with a set of words over the alphabet $\{d, f, p, z\}$ recognized by a **counter automaton**.

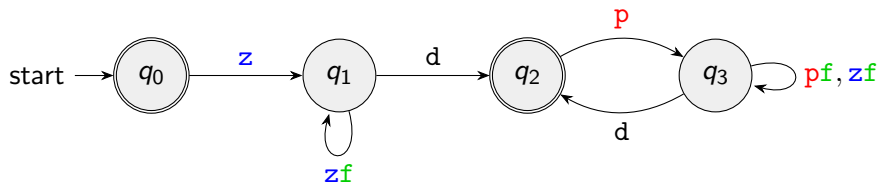


Theorem

$\mathcal{A}(P)$ has an algebraic generating function for any finite set of patterns P that contains 021.

Restricted growth functions avoiding 021

$\mathcal{R}(021)$ is in bijection with some ternary words recognized by the **finite automaton** below.



Theorem

$\mathcal{R}(P)$ has a **rational** generating function for any finite set of patterns P that contains 021.

Avoiding 021 and an increasing pattern

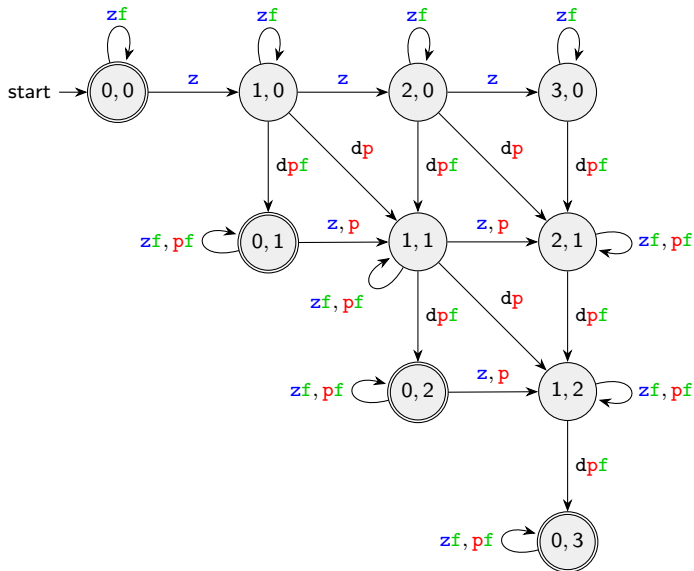
For a sequence in $\mathcal{I}(021)$, "avoiding the increasing pattern of length k " means the sequence can have at most $k - 1$ distinct values.

In order to turn these sequences into a rational language, we take an idea from Pantone: when placing the entry at position n in the sequence, we also decide whether the value n will appear in the future.

Theorem

$\mathcal{I}(P)$ has a rational generating function for any finite set of patterns P that contains 021 and an increasing pattern.

Automaton for $\mathcal{I}(021, 01234)$



Concluding remarks & open questions

- This approach could be applied to other (similar) types of sequences, or other definitions of patterns.
- We showed that some generating functions are always algebraic. How can we (efficiently) compute them? This is a work in progress.
- All results still hold when considering the pattern 10 instead of 021. Are there other patterns (or sets of patterns) like this?

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Thank you!